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Déformations des applications harmoniques tordues

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A Ester.

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Contents

Introduction

In this thesis, we propose a construction of a universal twisted harmonic map, together with its infinitesimal study up to the second order. We make use of these results to analyze the critical points and the positivity properties of the energy functional on the space of representations of a Kähler group.

Background and motivations

Corlette's theorem on twisted harmonic maps

Let (M, g) be a connected compact Riemannian manifold with a base point x_0, $\Gamma = \pi_1(M, x_0)$ its fundamental group, G a real connected algebraic group, and K a maximal compact subgroup. In this introduction, we are going to assume that G is semisimple, although reductive suffices for most results. Denote $N = G/K$ the associated Riemannian symmetric space of non-compact type and fix a representation $\rho\colon \Gamma \to G$. Since Eells and Sampson began the study of harmonic mappings between manifolds in [ES64], a great deal of work has been done on the subject. Notably for our discussion, the seminal paper by Corlette [Cor88] gave a necessary and sufficient condition for the existence of a harmonic ρ-equivariant mapping $f\colon \tilde{M} \to N$, where $\tilde{M}$ is the universal cover of M. Here, by "ρ-equivariant" we mean that it is equivariant with respect to the action of Γ on $\tilde{M}$ by deck transformations, and on G/K by isometries via ρ. Harmonicity can be characterized as critical points of the energy

$$E(f) = \frac{1}{2}\int_M \left\| \mathrm{d}f \right\|^2 \mathrm{dVol}_g,$$

where the norm is the product metric on $T\tilde{M}^* \otimes TN$ and the integral can be taken equivalently on any fundamental region of the covering. As in [ES64], another way to express harmonicity is the vanishing of the *tension field* $\tau(f)$, that in terms of a local frame $\{E_j\}$ can be expressed as

$$\tau(f) \stackrel{\text{loc}}{=} -\sum_{j,k} g^{jk} \nabla_{E_k} \mathrm{d}f(E_j),$$

where ∇ is the connection on f^*TN induced by the Levi-Civita connection on N and g^{jk} is the inverse matrix of the metric. Finally, denote by $\overline{\mathrm{Image}(\rho)}$ the real Zariski closure of the image $\rho(\Gamma) \subset G$, and by $H = Z_G(\overline{\mathrm{Image}(\rho)})$ its centralizer. Then the theorem of Corlette reads:

Theorem (Corlette)**.** *A harmonic ρ-equivariant map $f\colon \tilde{M} \to N$ exists if and only if $\overline{\mathrm{Image}(\rho)}_0$ is reductive. When it exists, it is unique, modulo multiplication on the left by an element of H.*

The multiplication by an element of H is not avoidable: If f is any such map, then for every $h \in H$, the map $\tilde{f} = h \cdot f$ is again harmonic (since H acts by isometries) and equivariant, as h commutes with every $\rho(\gamma)$, for $\gamma \in \Gamma$. Because of this lack of uniqueness, one cannot find a "universal twisted harmonic map" taking the form

$$\mathscr{H}\colon \mathbb{R}_{\mathrm{B}}(M, G)^{ss} \times \tilde{M} \to N, \quad \mathscr{H}(\rho, \cdot) \text{ is harmonic and } \rho\text{-equivariant.}$$

where $\mathbb{R}_{\mathrm{B}}(M, G) = \mathrm{Hom}(\Gamma, G)$ is the space of G-representations of Γ, and ss stands for "semisimple representations", that is, those such that $\overline{\mathrm{Image}(\rho)}$ is reductive. In the following, we will deal with this difficulty.

Hitchin's moduli space

In its groundbreaking paper [Hit87], Hitchin introduced and studied thoroughly the moduli space of solutions to the self-duality equation on a rank 2 vector bundle of odd degree on a Riemann surface $M = \Sigma$ of genus $g > 1$. This turns out to be very rich in structure, being a hyperkähler manifold of complex dimension $6g - 6$. Furthermore, it coincides with the moduli space of flat $\mathbb{P}\mathrm{SL}(2, \mathbb{C})$-connections and with that of connections A on a principal $\mathrm{SO}(3)$-bundle P together with a *Higgs field* Φ which, in the case of a Riemann surface, is just a $\bar{\partial}_A$-holomorphic $(1, 0)$-form with values in the Lie algebra bundle $\mathrm{ad}(P) \otimes \mathbb{C}$ (of course, all these object must be taken up to suitable isomorphism in order to obtain a finite dimensional moduli space).

This paper has generated a series of works aiming to replicate the same results, especially those regarding the topology of the moduli space, with more general groups G instead of $\mathbb{P}\mathrm{SL}(2, \mathbb{C})$. Among these, one can cite another paper by Hitchin [Hit92], dealing with $\mathrm{SL}(n, \mathbb{R})$ or, more generally, any split real form of a complex semisimple group, Gothen's thesis [Got95] and

García-Prada–Gothen–Mundet i Riera [GPGMiR13], where $G = \mathrm{Sp}(2n, \mathbb{R})$ is treated and Bradlow–García-Prada–Gothen [BGPG03] who have focused on $G = \mathrm{U}(p, q)$. The base idea of these works is to use the Morse theory applied to the function

$$\mu(A, \Phi) = 2i \int_\Sigma \operatorname{trace}(\Phi \wedge \Phi^*) = \|\Phi\|^2_{L^2},$$

which is a moment map for the S^1-action induced by $e^{i\theta}(A, \Phi) = (A, e^{i\theta}\Phi)$ and also a proper and pluri-subharmonic perfect Morse function. The study of this map, particularly of the positivity of the Hessian at its critical points, allows them to infer results on the topology of the spaces.

In fact, it turns out that, interpreting the (isomorphism classes of) flat G-connections as (conjugacy classes of) representations $\rho\colon \Gamma \to G$, the moment map above coincides, up to a constant multiple, with the energy of any harmonic ρ-equivariant map $f\colon \mathbf{H}^2 = \tilde{\Sigma} \to G/K$. Thus, the study of μ is a special case of a more general theory regarding the twisted harmonic mappings and their energy. In the following, we will attempt this more general work, proving and generalizing several of the properties of μ to more general manifolds M.

Again for Riemann surfaces, Toledo proved in a recent paper [Tol12] the pluri-subharmonicity of the energy of a twisted harmonic map in the different setting where one keeps the representation ρ fixed and allows the complex structure J to vary in the Teichmüller space. This work was also a source of inspiration for us. To complete the list, Biswas and Schumacher [BS06] have computed the complex hessian of the energy on moduli spaces of Higgs bundles in the general Kähler case (with possibly non-zero Chern classes), but allowing for stable bundles only.

Simpson's moduli spaces

Simpson [Sim92] generalized the correspondence between flat connections and "Higgs bundles" (holomorphic bundles carrying a Higgs field θ, that is a $(1,0)$-form satisfying the same requirements as Φ above, plus $\theta \wedge \theta = 0$) to higher dimensional compact Kähler manifolds X. Successively, in [Sim94], he constructed moduli spaces for these objects, this time for smooth projective varieties X; the stated correspondence gives a homeomorphism between the moduli spaces. The idea of the correspondence is as follows: Given a representation in a reductive algebraic group $\rho\colon \Gamma \to G \subseteq \mathrm{GL}(n, \mathbb{C})$ one has the flat bundle $(\mathcal{V}, D) = \tilde{X} \times_\Gamma \mathbb{C}^n$ consisting of the equivalence classes under the relation $(\tilde{x}, v) \cong (\gamma\tilde{x}, \rho(\gamma)v)$. A metric on this bundle corresponds to an equivariant family of positive definite hermitian matrices and, since

$GL(n,\mathbb{C})/U(n)$ classifies the positive definite hermitian matrices, a metric is just a ρ-equivariant map

$$f\colon \tilde{M} \to \mathrm{GL}(n,\mathbb{C})/U(n).$$

Then one can define a *harmonic metric* as a metric such that the corresponding map is harmonic. The conditions for existence of a harmonic metric are determined by Corlette's theorem, which also grants that, when such a metric exists, we can take it to be in G/K (remark that G/K is a totally geodesic subspace of $\mathrm{GL}(n,\mathbb{C})/U(n)$, so the notion of harmonicity is independent of the composition with the inclusion). Simpson proves that when $M = X$ is a Kähler manifold, a metric is harmonic if and only if it establishes a correspondence between flat bundles and Higgs bundles (with some stability condition and vanishing Chern classes), namely, that writing $D = \partial + \bar{\partial} + \beta$, where $\partial + \bar{\partial}$ is the metric-preserving part of D and β is self-adjoint, and letting θ be the $(1,0)$-part of β, the holomorphic bundle $\mathcal{E} = (\mathcal{V}, \bar{\partial})$ together with θ forms a Higgs bundle. This has several consequences; for example, the space of Higgs bundles is equipped with a natural $\mathbb{C}^*$-action (extending the S^1-action above) defined by

$$t \cdot (\mathcal{E}, \theta) = (\mathcal{E}, t\theta).$$

The fixed points of this action are known as complex variations of Hodge structure, which have been introduced by Deligne [Del87]. They are $\mathcal{C}^\infty$ flat bundles $(\mathcal{V}, D)$ with a decomposition $\mathcal{V} = \bigoplus_{r+s=w} \mathcal{V}^{r,s}$ and a flat hermitian form S such that D respects a "transversality" condition, the decomposition is S-orthogonal and S is definite of sign $(-1)^r$ on each $\mathcal{V}^{r,s}$ (see definition 1.9.1 for more details).

Associating to every representation ρ the energy of any ρ-equivariant harmonic map defines a non-negative function on the representation space $\mathbb{R}_{\mathrm{B}}(M,G)$ (which descends to the moduli space $\mathbb{M}_{\mathrm{B}}(X,G) = \mathbb{R}_{\mathrm{B}}(X,G)/\!\!/G$). When X is Kählerian, this coincides with the L^2-norm of the Higgs field θ, as in the case of a surface. The main applications of our results will be in studying the infinitesimal behavior of this functional.

Results of the thesis

Construction of the universal harmonic map and definitions

The main object of our discussion is a "universal twisted harmonic map". We discuss such an object in chapter 2; however, as mentioned above, we must introduce a new parameter in order to ensure uniqueness. Fix a base point $\tilde{x}_0$ in $\tilde{M}$. Since the non-uniqueness of the harmonic map is due to multiplication by an element of H, it is enough to fix the value of $f(\tilde{x}_0)$; because of this, we define the set Y as:

$$Y = \left\{ (n,\rho) \in N \times \mathbb{R}_{\mathrm{B}}(M,G) \;\middle|\; \begin{array}{l} \exists\, f\colon \tilde{M} \to N \ \rho\text{-equivariant and harmonic} \\ \text{such that } f(\tilde{x}_0) = n \end{array} \right\}.$$

Then, Corlette's theorem allows us to construct a well-defined universal map

$$\mathscr{H}\colon Y \times \tilde{M} \to N,$$

simply by denoting $\mathscr{H}(n,\rho,\cdot)$ the unique ρ-equivariant harmonic map such that $f(\tilde{x}_0) = n$. Writing $\mathbb{R}_{\mathrm{B}}(M,G) = \bigcup_i R_i$ for the decomposition into irreducible components and giving each R_i the reduced structure, we let U_i denote the open subset of the smooth part R_i^{sm} given by representations whose image is Zariski-dense in G. The main result regarding $\mathscr{H}$ is then the following:

Proposition. *The set $Y \subset N \times \mathbb{R}_{\mathrm{B}}(M,G)$ is closed. The universal map $\mathscr{H}\colon Y \times \tilde{M} \to N$ is continuous and its restriction to $Y \cap (N \times U_i) \times \tilde{M}$ is smooth.*

The proof of the generic smoothness is a simple adaptation of the original proof by Corlette; closedness of Y and continuity of $\mathscr{H}$ are obtained thanks to Arzelà-Ascoli theorem and an estimate on the derivatives of harmonic mappings. Since the energy is a continuous functional (with respect to the Sobolev $W^{1,2}$-norm) and thanks to an argument relating the energy of a representation to that of its semi-simplification, we get

Corollary. *The energy functional is continuous on the whole of $\mathbb{R}_{\mathrm{B}}(M,G)$. If U_i is not empty, the energy functional is smooth there.*

The remaining part of the thesis focuses on an infinitesimal study of the map $\mathscr{H}$. This relies on several technical results, which are gathered in chapter 1, where we introduce the notion of "polarized harmonic local systems".

Definition. A complex (resp. real) polarized harmonic local system is a harmonic bundle $(\mathcal{V}, D, f)$ with an involution σ and a flat hermitian (resp. symmetric or skew-symmetric) form S, such $S(\cdot,\sigma(\cdot))$ gives the metric f.

For such objects we introduce some further structure and prove general results. First of all, we define a "Maurer-Cartan" 1-form β with values in

$\mathrm{End}(\mathcal{V})$ as the pull-back of the right Maurer-Cartan form on N (extrinsecally, $\beta = \mathrm{d}f \cdot f^{-1}$; when $M = X$ is a Kähler manifold this is, in fact, the same β introduced above); this allows us to define the canonical connection as $\mathrm{d}^{\mathrm{can}} = \mathrm{d} - \beta$, which is thus metric, and to deduce a Weitzenböck formula for the codifferential d^*. Finally, we prove that the global sections of the local system are acted upon by σ (see corollary 1.6.9).

The main example of real polarized harmonic local system is the "adjoint" one on the symmetric space N, whose flat bundle is $N \times \mathfrak{g}$, with the identity map as harmonic metric, the Killing form as symmetric form and a Cartan involution as σ; in this case, the Maurer-Cartan form is in fact the usual one, acting through ad, and the canonical connection corresponds to the usual canonical connection on a Riemannian symmetric space through the identification

$$\begin{aligned} \vartheta_{TN} \colon N \times \mathfrak{g} &\to TN \\ (n, \xi) &\mapsto \frac{\partial}{\partial t}\big(\exp(t\xi) \cdot n\big)\Big|_{t=0}. \end{aligned}$$

We can construct other examples taking any ρ-equivariant harmonic map $f \colon \tilde{M} \to N$ and considering the pull-back of the adjoint local system, which actually lives on M. These are in fact the only examples to which we will apply our results.

This theory has special features when $M = X$ is a Kähler manifold. In this case, one also has functoriality of polarized harmonic local systems with respect to holomorphic maps (i.e. the pull-back through $\varphi \colon X \to X'$ of a polarized harmonic local system on X' gives one on X). Furthermore, complex (resp. real) variations of Hodge structure give complex (resp. real) polarized harmonic local systems, simply by disregarding the decomposition of $\mathcal{V}$ and keeping only the involution

$$\sigma = \sum_{r,s} (-1)^r \mathrm{Id}_{\mathcal{V}^{r,s}}.$$

Although this is *per se* a polarized harmonic local system $(\mathcal{V}, \sigma, S)$ via the harmonic metric f induced by the period mapping, in the following we will usually work with the structure obtained by pulling back the adjoint structure through f. It turns out that this the same as the "endomorphism polarized harmonic linear system" $\mathrm{End}(\mathcal{V}, \sigma, S)$.

First order analysis

The study of the first order of $\mathscr{H}$ at a given point coincides with the study of a first order deformation of a fixed harmonic ρ_0-equivariant map $f \colon \tilde{M} \to N$.

Definition. A first order deformation v of f is a section of f^*TN.

Since we want to allow the representation to change, we need to introduce the notion of the first order deformation of a representation:

Definition. A first order deformation $\rho_t^{(1)}$ of $\rho_0 \colon \Gamma \to G$ is a representation $\rho_t^{(1)} \colon \Gamma \to TG$ projecting to ρ_0.

Here, TG is given the structure of an algebraic group in any of the following equivalent ways: Either one sees G as the group of the real points of a connected algebraic group $\mathbb{G}$, and then $TG = \mathbb{G}(\mathbb{R}[\varepsilon]/(\varepsilon^2))$; or one can write an explicit bijection $TG = G \times \mathfrak{g}$, and put on the latter a natural group structure (cfr. [BS72]). In either way, a first order deformation of a representation is equivalent to the data of a 1-cocycle c for the adjoint action of Γ on $\mathfrak{g}$. Then, one can define equivariant deformations thanks to the natural action of TG on TN. Explicitly, this reads:

$$v(\tilde{x}) - \rho(\gamma)_* v(\gamma^{-1}\tilde{x}) = \vartheta_{TN}\big(f(\tilde{x}), c(\gamma)\big).$$

Finally, one has to deal with harmonicity. One calls a first order deformation v harmonic if it is a zero of the Jacobi operator, which has been introduced by Mazet in [Maz73]. In terms of a local frame $\{E_j\}$ this reads:

$$\mathcal{J}(v) = -\sum_{j,k} g^{jk}\Big(\nabla_{E_j}\nabla_{E_k} v + R^N\big(\mathrm{d}f(E_j), v\big)\mathrm{d}f(E_k)\Big) = 0,$$

where ∇ is the pull-back connection on f^*TN and R^N the curvature tensor on N.

Of course, the main examples of first order deformations are of the form $v = \frac{\partial f_t}{\partial t}\big|_{t=0}$, where $f_t \colon \tilde{M} \to N$ is a smooth family of smooth maps, for some real parameter $t \in (-\varepsilon, \varepsilon)$. Then one checks that if every f_t is harmonic, then v is, and if every f_t is ρ_t-equivariant for a family of representation ρ_t, then v is $\rho_t^{(1)}$-equivariant, where $\rho_t^{(1)}$ is determined by $c(\gamma) = \frac{\partial \rho_t(\gamma)}{\partial t}\big|_{t=0} \cdot \rho_0(\gamma)^{-1}$. The main result of chapter 3 is then a construction which gives all the first order $\rho_t^{(1)}$-equivariant and harmonic deformations. To state this, let $\omega \in \mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$ be the harmonic representative of the 1-cohomology class represented by c (here we are exploiting the usual isomorphism $H^1(\Gamma, \mathfrak{g}) \cong H^1(M, \mathrm{Ad}(\rho_0))$). Pulling back ω to $\tilde{M}$, we obtain a $\mathfrak{g}$-valued closed 1-form on $\tilde{M}$, which we can thus integrate to a map $F \colon \tilde{M} \to \mathfrak{g}$. One can further impose such an F to verify the equivariance relation:

$$F(\gamma\tilde{x}) = \mathrm{Ad}_{\rho_0(\gamma)} F(\tilde{x}) + c(\gamma).$$

We call such maps "$\rho_t^{(1)}$-equivariant of harmonic type"; one can prove that they always exist, and in fact form a torsor over $\mathfrak{h}$, the Lie algebra of H (hence, an affine space). The main theorem of chapter 3 is then:

Theorem A. *The natural map $\vartheta_{TN}\colon N \times \mathfrak{g} \to TN$ induces a surjective affine map*

$$\left\{ \begin{array}{c} F\colon \tilde{M} \to \mathfrak{g} \ : \rho_t^{(1)}\text{-equivariant} \\ \text{of harmonic type} \end{array} \right\} \xrightarrow{\vartheta_{TN}} \left\{ \begin{array}{c} v \in \mathcal{C}^\infty(f^*TN) \text{ harmonic} \\ \text{and } \rho_t\text{-equivariant.} \end{array} \right\}$$

On Kähler manifolds, the construction is functorial.

In chapter 4 we apply this result to compute the first variation of the energy of a family of harmonic and equivariant maps. The formula (given in proposition 4.1.3) reads:

$$\frac{\partial E_t}{\partial t}\bigg|_{t=0} = \int_M \langle \omega, \beta \rangle \mathrm{dVol}_g.$$

Here, $\frac{\partial E_t}{\partial t}$ is just a notation; it can be seen as the the actual derivative of the energy of any smooth family f_t extending (f, v), which is supposed to be $\rho_t^{(1)}$-equivariant and harmonic. Thanks to this result, we are able to investigate the critical points of the energy. Since the spaces we work on are not smooth, we actually *define* a critical point as one for which $\int_M \langle \omega, \beta \rangle = 0$ for every $\omega \in \mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$. The generic smoothness of the map $\mathscr{H}$ ensures that this coincides with the usual notion at smooth points. Then we prove:

Theorem B. *Let $M = X$ be a compact Kähler manifold. Then the critical points of the energy functional on $\mathbb{R}_{\mathrm{B}}(X, G)$ are exactly the representations coming from complex variations of Hodge structure.*

The proof that every complex variation of Hodge structure is a critical point is just the observation that, in this case, β can be written as $(D'' - D')\gamma$, for some section γ of $\tilde{X} \times \mathfrak{g}$, and then, thanks to the Kähler identities, one uses that ω is both D''- and D'-harmonic. For the converse, one investigates the special deformation $(\mathcal{E}, t\theta)$, proving that the variation of the energy in this direction coincides with $\|\omega\|_{L^2}^2$. Thus, in a critical point one must have $\omega = 0$, which links to θ being zero in Dolbeault cohomology and in turn to $(\mathcal{E}, \theta)$ being a complex variation of Hodge structure. This proof also gives the following result, which is actually a restatement of the theorem:

Corollary. *If $M = X$ is Kähler, the gradient of the energy, defined with respect to the Weil-Petersson metric, is (twice) the vector field tangent to the $\mathbb{C}^*$-action.*

In particular, this gives an "infinitesimal" substitute for the $\mathbb{C}^*$-action in the general Riemannian case.

Remark that when $X = \Sigma$ is a Riemann surface, this result was already in [Hit87]: Since the energy is the moment map for the circle action, a point is critical if and only if it is fixed; but being fixed by S^1 or by the whole of $\mathbb{C}^*$ is, in fact, equivalent. This fact allows one to deduce easily theorem B in the case where X is projective and the Kähler class is integral and very ample; thus the theorem is really new only when X is just supposed to be Kähler (or projective with a non integral Kähler class).

Second order analysis

For the study of the second order analysis we proceed along the same steps as for the first order case. However, definitions are less natural here, and some obstruction arises.

A second order deformation of a map $f\colon \tilde{M} \to N$ can be defined either as a section of $f^* J^2 N$, the pull-back of the second jet bundle, or as a pair (v, w) of sections of f^*TN. The two constructions are related by the canonical connection, which gives an isomorphism of the two bundles. We have chosen the second approach; to fix the ideas, for a smooth family $f_t\colon \tilde{M} \to N$, the section w is defined by

$$w = \frac{D}{\partial t}\frac{\partial f_t}{\partial t}\Big|_{t=0},$$

where $\frac{D}{\partial t} = \nabla_{\frac{\partial}{\partial t}}$ is the covariant derivative along $\frac{\partial f_t}{\partial t}\big|_{t=0}$. As in the first order case, one can start with a family of ρ_t-equivariant (resp. harmonic) maps, and deduce formulas for $\rho_t^{(2)}$-equivariant (resp. harmonic) second order deformations. Precise definitions can be found in section 5.1, here we remark only that the condition of harmonicity is again expressed in terms of the Jacobi operator $\mathcal{J}$ and the Riemann curvature tensor R^N, and that again a second order deformation $\rho_t^{(2)}$ of $\rho_t^{(1)}\colon \Gamma \to TG$ is the same thing as a map $k\colon \Gamma \to \mathfrak{g}$ making the pair (c, k) a 1-cocycle for the adjoint action $\mathrm{Ad}(\rho_t^{(1)})$ of Γ on $\mathfrak{g} \times \mathfrak{g} \cong \mathfrak{g} \otimes \mathbb{R}[t]/(t^2)$.

Then one proceeds to define $\rho_t^{(2)}$-equivariant maps $(F, F_2)\colon \tilde{M} \to \mathfrak{g} \times \mathfrak{g}$ of harmonic type. The definition for equivariance is just in the same spirit as the first order: One requires that

$$(F, F_2)(\gamma\tilde{x}) = \rho_t^{(1)}(\gamma) \cdot \big(F(\tilde{x}), F_2(\tilde{x})\big) + (c(\gamma), k(\gamma)).$$

On the other hand, harmonicity is more complicated: The 1-form $\mathrm{d}F_2$ is not even a section of $\mathrm{Ad}(\rho_0)$, so we modify it defining the 1-form ψ as:

$$\psi = \mathrm{d}F_2 + [\omega, F].$$

This 1-form is $\mathrm{Ad}(\rho_0)$-valued, but neither closed nor co-closed: We say that (F, F_2) is of harmonic type if ψ satisfies the following equations (in terms of a local orthonormal frame $\{E_j\}$):

$$\mathrm{d}\psi = -[\omega, \omega]; \quad \mathrm{d}^*\psi \overset{\mathrm{loc}}{=} \sum_j \big[\sigma(\omega(E_j)), \omega(E_j)\big], \tag{5.10}$$

where σ is the involution of the polarized harmonic local system on $\tilde{M} \times \mathfrak{g}$. Then one can define a map $\vartheta_{J^2N}\colon N \times \mathfrak{g} \times \mathfrak{g} \to TN \times_N TN$, by:

$$\vartheta_{J^2N}(n, \xi, \mu) = \Big(\vartheta_{TN}(n, \xi), \vartheta_{TN}\Big(n, \mu + \big[\xi^{[\mathfrak{k}]}, \xi^{[\mathfrak{p}]}\big]\Big)\Big).$$

Here, $[\mathfrak{k}] = [\mathfrak{k}]_n$ (resp. $[\mathfrak{p}]$) designates the projection to the 1-eigenspace (resp. the -1-eigenspace) of σ. In particular, this map is no longer linear, but one can prove nevertheless that it sends $\rho_t^{(2)}$-equivariant (F, F_2) of harmonic type to $\rho_t^{(2)}$-equivariant and harmonic second order deformations.

In this way, we have again a way to construct second order harmonic and $\rho_t^{(2)}$-equivariant deformations. However, unlike the first order case, here there are obstructions to the existence of these objects. We collect the main results in the next theorem.

Theorem C. *Let $\rho_t^{(2)}\colon \Gamma \to J^2G$ be a second order deformation of $\rho_t^{(1)} = (\rho_0, c)$ and $f\colon \tilde{M} \to N$ be harmonic and ρ_0-equivariant. The existence of a 1-form ψ satisfying* (5.10) *implies that every first order $\rho_t^{(1)}$-equivariant and harmonic (f, v) extends to a second order $\rho_t^{(2)}$-equivariant and harmonic (f, v, w). Furthermore:*

- *If G is complex, there is a converse: Such a ψ exists if, and only if, we can find second order harmonic deformations (f, v, w) and $(f, \tilde{v}, \tilde{w})$ equivariant one for (ρ_0, c) and the other for (ρ_0, ic) (the second orders of the representations are not significant).*

- *The 1-form ψ exists if, and only if, ω is a critical point for the L^2-norm in its orbit under the adjoint action of $H = Z_G(\mathrm{Image}(\rho_0))$ on $\mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$. Hence, if G is complex, given (ρ_0, c) there exists a metric f for which ψ exists if, and only if, ω is polystable for this action.*

- *A sufficient condition for the existence of ψ for every metric f is $H^1(M, \mathrm{Ad}(\rho_t^{(1)}))$ being a flat $\mathbb{R}[t]/(t^2)$-module. When G is complex, this is also necessary.*

We give examples of $\rho_t^{(1)} = (\rho_0, c)$ for which there is no metric f admitting a second order deformation (f, v, w) which is both harmonic and $\rho_t^{(2)}$-equivariant (again, for any $\rho_t^{(2)}$ extending $\rho_t^{(1)}$). However, we currently know of no example for which (ρ_0, c) admits such a metric but (ρ_0, ic) does not, that is, an example of a deformable f for which ψ does not exist.

Under the assumption of the existence of ψ, we can give a formula for the second variation of the energy similar to that we gave for the first variation:

$$\frac{\partial^2 E_t}{\partial t^2}\Big|_{t=0} = \int_M \Big(\big\langle \psi, \beta \big\rangle + \|\omega^{[\mathfrak{p}]}\|^2\Big) \mathrm{dVol}_g.$$

With this result at hand, we can prove the following:

Theorem D. *If G is a complex group, the energy is a strictly pluri subharmonic function on the smooth points of the moduli space. Consequently, it gives a pluri-subharmonic function on its normalization.*

As a matter of fact, the energy turns out to be a Kähler potential with respect to the Betti complex structure and the Weil-Petersson metric.

Finally, in the Kählerian case, we are able to study the positivity of the Hessian of the energy at the critical points, in terms of the eigenvalues of the local generator of the S^1-action, in the same spirit as [Hit92]. To stress this parallelism, we introduce the following notation: For every $\mathfrak{g}$-valued 1-form α, denote $\alpha^{[\mathfrak{k}]}$ (resp. $\alpha^{[\mathfrak{p}]}$) the 1-form such that, for every (real) tangent field $\chi \in \Xi(X)$, exponents $[\mathfrak{k}]$ and $[\mathfrak{p}]$ denote projection on the corresponding subspaces:

$$\alpha^{[\mathfrak{k}]}(\chi) = \alpha(\chi)^{[\mathfrak{k}]} \in [\mathfrak{k}] = \mathrm{Ker}(\sigma - \mathrm{Id}), \quad \alpha^{[\mathfrak{p}]}(\chi) = \alpha(\chi)^{[\mathfrak{p}]} \in [\mathfrak{p}] = \mathrm{Ker}(\sigma + \mathrm{Id}).$$

Furthermore, let α' (resp. α'') denote the holomorphic (resp. anti-holomorphic) part of α. Then define $\dot{A} = (\omega^{[\mathfrak{k}]})''$ and $\dot{\Phi} = (\omega^{[\mathfrak{p}]})'$ (see [Hit87] for a more direct interpretation of these symbols, in the case of a Riemann surface $X = \Sigma$).

Theorem E. *Let G be complex, $\rho_0 \colon \Gamma \to G$ be induced by a complex variation of Hodge structures and $f \colon \tilde{X} \to N$ the harmonic map induced by the period mapping. Then, the second order variation of the energy along $\omega \in \mathcal{H}^1(X, \mathrm{Ad}(\rho_0))$ can be written as*

$$\frac{\partial^2 E_t}{\partial t^2}\Big|_{t=0} = 2\int_X \Big(\sum_p -p\big\|\dot{A}^{-p,p}\big\|^2 + (1-p)\big\|\dot{\Phi}^{-p,p}\big\|^2\Big) \mathrm{dVol}_g,$$

where $\xi = \sum_p \xi^{-p,p}$, for $\xi \in \mathfrak{g}$, denotes the Hodge decomposition of $\mathfrak{g}$ induced by the complex variation of Hodge structure. In the special case where ω

takes values in the Lie algebra of the real Zariski closure of $\mathrm{Image}(\rho_0)$*, this expression may be rewritten in terms of the Deligne's Hodge structure* $\omega = \sum_{P+Q=1} \omega^{(P,Q)}$ *as*

$$\frac{\partial^2 E_t}{\partial t^2}\Big|_{t=0} = \int_X \bigg(\sum_{P\equiv 0} P \big\| \omega^{(P,Q)} \big\|^2 + \sum_{P\equiv 1} Q \big\| \omega^{(P,Q)} \big\|^2 \bigg) \mathrm{dVol}_g.$$

Corollary. *If the representation* ρ_0 *is induced by a complex variation of Hodge structure whose period domain is of Hermitian symmetric type, then the Hessian of the energy at* ρ_0 *is semi-positive definite along all deformations* $\rho_t \colon \Gamma \to G_0$*, where* G_0 *denotes the real Zariski closure of* $\mathrm{Image}(\rho_0)$*.*

A preprint based on the topics of this thesis is available at [Spi13].

Organization of the thesis

Chapter 1 is devoted to background material and technical results about polarized harmonic local systems. The definition is given in section 1.2, and the main example is discussed in section 1.3. After discussing briefly the structure of the tangent bundle of a Riemannian symmetric space of non-compact type, the main results about polarized harmonic local systems are given in sections 1.5 and 1.6. In sections 1.7 and 1.8 we discuss the special cases $M = S^1$ and $M = X$ Kähler, and the notion of complex variation of Hodge structure is reviewed in section 1.9.

In chapter 2 we extend Corlette's theorem to the construction of a "universal twisted harmonic map". We prove it to be continuous in section 2.2; with some additional work, we can then prove the continuity of the energy on the whole of $\mathbb{R}_{\mathrm{B}}(M, G)$.

Chapter 3 is devoted to the study of the first order deformations of harmonic maps, aiming to prove theorem A. Equivariant first order deformations are defined and discussed in section 3.2, and harmonic ones in section 3.3. In section 3.4 we show how the constructive process of theorem A works, and we prove that this gives all equivariant and harmonic deformations in section 3.5. Finally, we give a precise statement for theorem A in section 3.6, and in section 3.7 we analyze the results when $G = \mathrm{GL}(1, \mathbb{C})$ (in which case non-abelian cohomology reduces to the usual one).

The results thus obtained are applied in chapter 4 to the study of the first variation of the energy functional, which starts by proving the formula in terms of the scalar product of ω and β mentioned above. This allows us to prove that complex variations of Hodge structure are critical points of the energy: This is done in section 4.2. In section 4.3 we prove the converse, by analyzing the variation of the energy under the $\mathbb{C}^*$-action.

The second order analysis is carried through in chapter 5, which is devoted to the proof of theorem C. We first give all the definitions and the matchings between them in sections 5.1 and 5.2. Then we introduce the action of H on $\mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$, in terms of which we construct $\rho_t^{(2)}$-equivariant maps (F, F_2) of harmonic type. This is done in sections 5.3 and 5.4. The link between existence of (F, F_2) and that of a deformation (v, w) of f is analyzed in section 5.5. Section 5.6 is devoted to the example of a representation $\rho_t^{(1)}$ for which there exists no second-order deformable metric, and in section 5.8 we prove that when one has a smooth family of maps f_t, for $t \in (-\varepsilon, \varepsilon)$, the definitions we gave coincide with the natural ones. Remark that what we denoted with theorem C in this introduction is split into proposition 5.4.6, proposition 5.5.5, theorem 5.7.1 and proposition 5.7.2.

Finally, in chapter 6, we apply these results to give a formula for the second variation of the energy in terms of ω and ψ (under the assumption that this exists). This allows us to prove, on the one hand, the pluri subharmonicity of the energy of theorem D, which is done in section 6.2 and, on the other hand, the formula of theorem E. This last part is the content of section 6.3.

Introduction (version française)

Dans cette thèse, on propose une construction d'une application harmonique tordue universelle et son étude infinitésimale jusqu'à l'ordre deux. On utilise ces résultats pour analyser les points critiques et les propriétés de positivité de la fonctionnelle de l'énergie sur l'espace des représentations d'un groupe de Kähler.

Contexte et motivations

Le théorème de Corlette sur les applications harmoniques tordues

Soit (M, G) une variété riemannienne compacte et connexe avec un point base x_0; soit $\Gamma = \pi_1(M, x_0)$ son groupe fondamental, G un groupe algébrique réel connexe et K un de ses sous-groupes compacts maximaux. Dans cette introduction, on va supposer G semi-simple, même si pour la plupart des résultats réductif suffirait. Notons $N = G/K$ l'espace riemannien symétrique du type non-compacte associé, et fixons une représentation $\rho\colon \Gamma \to G$. Depuis le début de l'étude des applications harmoniques par Eells et Sampson ([ES64]), une quantité importante de travail a été faite dans ce sujet. D'importance pour notre discussion est le papier fondateur de Corlette [Cor88], qui a donné une condition nécessaire et suffisante pour l'existence d'une application ρ-équivariante et harmonique $f\colon \tilde{M} \to N$, où $\tilde{M}$ dénote le revêtement universel de M. Ici, "ρ_0-équivariant" signifie qu'elle est équivariante par rapport à l'action naturelle de Γ sur $\tilde{M}$ et par isométries sur $N = G/K$ via ρ. L'harmonicité peut être caractérisée comme l'être un point critique de l'énergie

$$E(f) = \frac{1}{2}\int_M \left\|\mathrm{d}f\right\|^2 \mathrm{dVol}_g,$$

où la norme est induite par la métrique produit sur $T\tilde{M}^* \otimes TN$ et l'intégrale est prise de façon équivalente sur n'importe quel domaine fondamental du revêtement. Comme est déjà noté dans [ES64], être harmonique équivaut à l'annulation du *champs de tension* $\tau(f)$, qui est l'équation d'Euler-Lagrange du problème variationnel et en termes d'un repère orthonormé local $\{E_j\}$ peut s'exprimer par

$$\tau(f) \overset{\text{loc}}{=} -\sum_{j,k} g^{jk} \nabla_{E_k} \mathrm{d}f(E_j),$$

où ∇ est la connexion sur f^*TN induite par la connexion de Levi-Civita sur N et g^{jk} est la matrice inverse de la métrique. Finalement, notons par $\overline{\text{Image}(\rho)}$ l'adhérence de Zariski de l'image $\rho(\Gamma) \subset G$, et par $H = Z_G(\overline{\text{Image}(\rho)})$ son centralisateur. Alors, le théorème de Corlette affirme:

Théorème (Corlette)**.** *Une application harmonique et ρ-équivariant $f\colon \tilde{M} \to N$ existe si et seulement si $\overline{\text{Image}(\rho)}_0$ est un sous-groupe réductif. Quand elle existe, elle est unique, à multiplication par un élément $h \in H$ près.*

La multiplication par un élément de H n'est pas évitable: si f est une application harmonique tordue, alors pour chaque $h \in H$ l'application $\tilde{f} = h \cdot f$ est encore harmonique, car H agit par isométries, et équivariante, car h commute avec $\rho(\gamma)$ pour tout $\gamma \in \Gamma$. À cause de ce manque d'unicité, on ne peut pas trouver une "application harmonique tordue universelle" de la forme

$$\mathscr{H}\colon \mathbb{R}_{\mathrm{B}}(M,G)^{ss} \times \tilde{M} \to N, \quad \mathscr{H}(\rho,\cdot) \text{ est harmonique et } \rho\text{-équivariante.}$$

où $\mathbb{R}_{\mathrm{B}}(M,G) = \mathrm{Hom}(\Gamma, G)$ est l'espace des G-représentations de Γ et *ss* signifie "représentations semi-simples", c'est-à-dire, telles que $\overline{\text{Image}(\rho)}$ est réductif. Dans la suite, on s'occupera de cette difficulté.

L'espace des modules de Hitchin

Dans son célèbre papier [Hit87], Hitchin a introduit et étudié minutieusement l'espace des modules des solutions aux équations d'auto-dualité sur un fibré vectoriel de rang 2 au-dessus d'une surface de Riemann $M = \Sigma$ de genre $g > 1$. Cet espace se révèle être muni d'une structure très riche, étant une variété hyperkählerienne lisse de dimension complexe $6g - 6$. De plus, il coïncide avec l'espace de modules des $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$-connexions plates et avec celui des connexions A sur un $\mathrm{SO}(3)$-fibré principal P avec un *champs de Higgs* Φ qui, dans le cas d'une surface de Riemann, est simplement une forme $(1,0)$, holomorphe par rapport à $\bar{\partial}_A$, à valeurs dans le fibré $\mathrm{ad}(P) \otimes \mathbb{C}$ (bien sûr, tous ces objets sont pris à isomorphismes près, pour obtenir des espaces de modules de dimension finie).

Cet article a engendré une suite de travaux essayant de reproduire des résultats similaires, notamment ceux regardants la topologie de l'espace de modules, avec un groupe plus général G à la place de $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$. Parmi ceux, on peut citer un autre article de Hitchin [Hit92], qui traite le cas $G = \mathrm{SL}(n,\mathbb{R})$ ou, plus généralement, n'importe quelle forme réelle "split" d'un groupe semi-simple complexe; la thèse de Gothen [Got95] et García-Prada–Gothen–Mundet i Riera [GPGMiR13], où $G = \mathrm{Sp}(2n,\mathbb{R})$; et Bradlow–García-Prada–Gothen [BGPG03], qui se sont concentrés sur $G = \mathrm{U}(p,q)$. L'idée de base de ces travaux est d'utiliser la théorie de Morse appliquée à la fonction

$$\mu(A,\Phi) = 2i\int_\Sigma \mathrm{trace}(\Phi\wedge\Phi^*) = \|\Phi\|^2_{L^2},$$

qui est une application des moments pour l'action de S^1 induite par $e^{i\theta}(A,\Phi) = (A, e^{i\theta}\Phi)$ et aussi une fonction de Morse parfaite, propre et pluri sous-harmonique. L'étude de cette application, en particulier de la positivité de la matrice hessienne dans ses points critiques, a leur permis d'obtenir des résultats sur la topologie de ces espaces.

Il s'avère en fait que, en interprétant les (classes d'isomorphisme des) G-connexions plates comme (classes de conjugaison de) représentations $\rho\colon\Gamma\to G$, l'application des moments ci-dessus coïncide, à un multiple constant près, avec l'énergie de n'importe quelle application $f\colon \mathbf{H}^2 = \tilde{\Sigma}\to G/K$ harmonique et ρ-équivariante . Par conséquent, l'étude de μ est un cas particulier d'une théorie plus générale concernante les applications harmoniques tordues et leurs énergie. Dans la suite, on visera cet étude plus générale, démontrant et généralisant plusieurs parmi les propriétés de μ pour des variétés M plus générales.

Pour des surfaces de Riemann, Toledo a démontré dans un travail récent [Tol12] la pluri sous-harmonicité de l'énergie d'une application harmonique tordue dans le cadre différent où l'on fixe la représentation ρ et varie la structure complexe J dans l'espace de Teichmüller. Ce travail a aussi été une source d'inspiration.

Les espaces de modules de Simpson

Simpson [Sim92] a généralisé la correspondance entre les connexions plates et le "fibrés de Higgs" (fibrés holomorphes avec un champ de Higgs θ, qui est une forme de type $(1,0)$ avec les mêmes propriétés que Φ ci-dessus, et aussi telle que $\theta\wedge\theta = 0$) à des variétés kähleriennes compactes lisse X de dimension supérieure. Dans un second moment [Sim94], il a construit des espaces de modules pour ces objets, cette fois pour des variétés X projectives lisses; la correspondance qu'on vient de mentionner induit alors un homéomorphisme entre les espaces de modules. L'idée de la correspondance est: donnée une représentation dans un groupe réductif $\rho\colon \Gamma \to G \subseteq \mathrm{GL}(n,\mathbb{C})$, on a le fibré plat $(\mathcal{V}, D) = \tilde{X} \times_\Gamma \mathbb{C}^n$ formé des classes d'équivalence sous la relation $(\tilde{x}, v) \cong (\gamma\tilde{x}, \rho(\gamma)v)$. Une métrique sur ce fibré correspond à une famille équivariante de matrices définies positives et, comme $\mathrm{GL}(n,\mathbb{C})/U(n)$ classifie les matrices hermitiennes définies positives, une métrique est une application ρ-équivariante:

$$f\colon \tilde{M} \to \mathrm{GL}(n,\mathbb{C})/U(n).$$

On peut alors définir une *métrique harmonique* comme une métrique telle que l'application correspondante est harmonique. Les conditions pour l'existence d'une métrique harmonique sont déterminées par le théorème de Corlette, qui assure aussi que, quand cette métrique existe, on peut la prendre à valeurs dans G/K (remarquons que G/K est un sous-espace totalement géodésique de $\mathrm{GL}(n,\mathbb{C})/U(n)$, donc la notion d'harmonicité est indépendante de la composition avec l'inclusion). Simpson démontre que quand $M = X$ est une variété de Kähler, une métrique est harmonique si et seulement si elle établit une correspondance entre fibrés plats et fibrés de Higgs (avec une condition de stabilité et classes de Chern nulles). Explicitement, en écrivant $D = \partial + \bar{\partial} + \beta$, où $\partial + \bar{\partial}$ est la partie qui préserve la métrique et β est auto-adjointe, et définissant θ la partie $(1,0)$ de β, le fibré holomorphe $\mathcal{E} = (\mathcal{V}, \bar{\partial})$ avec θ forme un fibré de Higgs. Ceci a plusieurs conséquences; par exemple, l'espace des fibrés de Higgs est muni d'une action naturelle de $\mathbb{C}^*$ (qui étend l'action de S^1 définie auparavant) par

$$t \cdot (\mathcal{E}, \theta) = (\mathcal{E}, t\theta).$$

Les points fixes de cet action sont connus comme "variations complexes de structures de Hodge", introduites par Deligne [Del87]. Ils sont des fibrés $\mathcal{C}^\infty$ plats $(\mathcal{V}, D)$ avec une décomposition lisse $\mathcal{V} = \bigoplus_{r+s=w} \mathcal{V}^{r,s}$ et une forme hermitienne parallèle S telles que D vérifie une condition de "transversalité" par rapport à la décomposition, que la décomposition est orthogonale par rapport à S et que S est définie de signe $(-1)^r$ sur chaque $\mathcal{V}^{r,s}$ (voir Définition 1.9.1 pour plus de détails).

Associer à chaque représentation ρ l'énergie d'une application harmonique ρ-équivariante donne une fonction non-négative sur l'espace des représentations $\mathbb{R}_{\mathrm{B}}(M,G)$ (qui descend à l'espace de modules $\mathbb{M}_{\mathrm{B}}(X,G) = \mathbb{R}_{\mathrm{B}}(X,G)/\!\!/G$). Quand X est Kähler, ceci coïncide avec la norme L^2 du champ de Higgs θ, comme pour le cas des surfaces. Les applications principales de nos résultats seront dans l'étude du comportement infinitésimal de cette fonctionnelle. Pour terminer la liste, Biswas and Schumacher [BS06] ont calculé l'hessienne complexe de l'énergie sur les espaces des modules des fibrés de Higgs dans le cas kählerien (avec classes de Chern éventuellement non-nulles), mais en ne considérant que des fibrés de Higgs stables.

Résultats de la thèse

Construction de l'application harmonique tordue universelle et définitions

L'objet principal de notre discussion est une "application harmonique tordue universelle". On examine cet objet dans le chapitre 2; par contre, on a mentionné ci-dessus qu'on est obligé d'introduire un nouveau paramètre pour assurer l'unicité. Fixons un point base $\tilde{x}_0$ dans $\tilde{M}$. Comme la non-unicité de l'application harmonique est due à une multiplication par un élément de H, il suffit de fixer la valeur de $f(\tilde{x}_0)$; à cause de cela, on définit un ensemble Y par:

$$Y = \left\{ (n, \rho) \in N \times \mathbb{R}_{\mathrm{B}}(M, G) \,\middle|\, \begin{array}{l} \exists\, f \colon \tilde{M} \to N \ \rho\text{-équivariante et harmonique} \\ \text{telle que } f(\tilde{x}_0) = n \end{array} \right\}.$$

Alors, le théorème de Corlette nous permet de construire une application universelle bien définie

$$\mathscr{H} \colon Y \times \tilde{M} \to N,$$

notant simplement par $\mathscr{H}(n, \rho, \cdot)$ l'unique application harmonique et ρ-équivariante telle que $f(\tilde{x}_0) = n$. En notant $\mathbb{R}_{\mathrm{B}}(M, G) = \bigcup_i R_i$ la décomposition dans des composantes irréductibles et donnant à chaque R_i la structure réduite, on pose U_i le sous-ensemble ouvert des points lisses R_i^{sm} formé par les représentations dont l'image est Zariski-dense dans G. Le résultat principal sur $\mathscr{H}$ est alors le suivant:

Proposition. *L'ensemble $Y \subset N \times \mathbb{R}_{\mathrm{B}}(M, G)$ est fermé. L'application universelle $\mathscr{H} \colon Y \times \tilde{M} \to N$ est continue et sa restriction à $Y \cap (N \times U_i) \times \tilde{M}$ est lisse.*

La preuve de la lissité générique est une simple adaptation de la preuve originale du théorème de Corlette; l'être fermé de Y et la continuité de $\mathscr{H}$ sont obtenus grâce au théorème de Arzelà-Ascoli et une estimation sur les dérivées des applications harmoniques. Puisque l'énergie est une fonctionnelle continue (par rapport à la norme de Sobolev $W^{1,2}$) et grâce à un argument qui relie l'énergie d'une représentation à celle de sa semi-simplification, on obtient:

Corollaire. *La fonctionnelle de l'énergie est continue sur* $\mathbb{R}_{\mathrm{B}}(M, G)$. *Si* U_i *n'est pas vide, l'énergie est lisse sur cet ouvert.*

La partie restante de la thèse se focalise sur une étude infinitésimale de l'application $\mathscr{H}$. Ceci repose sur plusieurs résultats techniques, qui nous collectionnons dans le chapitre 1, où on introduit la notion de "système local harmonique polarisé".

Définition. Un système local harmonique polarisé complexe (resp. réel) est un fibré harmonique $(\mathcal{V}, D, f)$ avec une involution σ et une forme quadratique plate hermitienne (resp. symétrique) S telle que $S(\cdot, \sigma(\cdot))$ donne la métrique f.

On discute de la structure de ces objets et de certains résultats généraux sur eux. Premièrement, on définit une 1-forme de "Maurer-Cartan", β, à valeurs dans $\mathrm{End}(\mathcal{V})$, comme le pull-back de la forme de Maurer-Cartan droite sur N (de façon extrinsèque, $\beta = \mathrm{d}f \cdot f^{-1}$; quand $M = X$ est kählerienne, c'est en fait la même β qu'on a introduit plus haut); ceci nous permet de définir la connexion canonique par $\mathrm{d}^{\mathrm{can}} = \mathrm{d} - \beta$, qui est ainsi une connexion métrique et de déduire une formule de Weitzenböck pour la codifferentielle d^*. Finalement, on démontre que les sections globales du système local sont invariantes par σ (cfr. Corollaire 1.6.9).

L'exemple principal de système local harmonique polarisé réel est celui du système "adjoint" sur un espace symétrique N, dont le fibré plat est $N \times \mathfrak{g}$, la métrique harmonique est donnée par l'identité, la forme symétrique par la forme de Killing et σ par une involution de Cartan. Dans ce cas, la forme de Maurer-Cartan est l'usuelle, qui agit par ad, et la connexion canonique correspond à la connexion canonique usuelle sur un espace riemannien symétrique par l'identification

$$\begin{aligned} \vartheta_{TN} \colon N \times \mathfrak{g} &\to TN \\ (n, \xi) &\mapsto \frac{\partial}{\partial t}\big(\exp(t\xi) \cdot n\big)\Big|_{t=0}. \end{aligned}$$

On construit d'autres exemples en considérant une application harmonique ρ-équivariante $f \colon \tilde{M} \to N$ et prenant le pull-back du système local adjoint, qui, de plus, peut s'identifier à un système local sur M. En fait, tous les exemples qu'on considère sur les chapitres suivants seront de cette forme.

Cette théorie a des propriétés meilleures quand $M = X$ est une variété kählerienne. Dans ce cas, on a aussi la fonctorialité des systèmes locaux harmoniques polarisés par rapport aux applications holomorphes (c'est-à-dire, le pull-back par $\varphi \colon X \to X'$ d'un système local harmonique polarisé sur

X' en donne un autre sur X). De plus, les variations complexes de structures de Hodge (resp. réelles) donnent des systèmes locaux harmoniques polarisés complexes (resp. réels), simplement en oubliant la décomposition de $\mathcal{V}$ et gardant uniquement l'involution

$$\sigma = \sum_{r,s} (-1)^r \mathrm{Id}_{\mathcal{V}^{r,s}}.$$

Même si cet objet donne un système local harmonique polarisé $(\mathcal{V}, \sigma, S)$ grâce à la métrique harmonique f induite par l'application de périodes, dans la suite on travaillera d'habitude avec la structure obtenue par pull-back via f de la structure adjoint. Il s'avère alors que ceci est la même chose que le "système local harmonique polarisé endomorphisme", $\mathrm{End}(\mathcal{V}, \sigma, S)$.

Analyse du premier ordre

L'étude du premier ordre de $\mathscr{H}$ dans un point donné coïncide avec l'étude d'une déformation au premier ordre d'une application harmonique ρ_0-équivariante donnée $f\colon \tilde{M} \to N$.

Définition. Une déformation au premier ordre v de f est une section de f^*TN.

Puisqu'on veut permettre des changements dans la représentation, on doit introduire la notion de déformations au premier ordre des représentations:

Définition. Une déformation au premier ordre $\rho_t^{(1)}$ de $\rho_0\colon \Gamma \to G$ est une représentation $\rho_t^{(1)}\colon \Gamma \to TG$ qui est envoyée sur ρ_0 par $TG \to G$.

Ici, TG a une structure de groupe algébrique dans une des manières équivalentes: soit on voit G comme le groupe des points réels d'un groupe algébrique connexe $\mathbb{G}$, et alors $TG = \mathbb{G}(\mathbb{R}[\varepsilon]/(\varepsilon^2))$; soit on écrit de façon explicite une bijection $TG = G \times \mathfrak{g}$ et on met ensuite une structure de groupe naturelle sur le côté droit (cfr. [BS72]). Dans les deux cas, une déformation au premier ordre d'une représentation est équivalente à considérer un 1-cocycle c pour l'action adjointe de Γ sur $\mathfrak{g}$. Alors, on peut définir les déformations équivariantes grâce à l'action naturelle de TG sur TN. Explicitement, on obtient:

$$v(\tilde{x}) - \rho(\gamma)_* v(\gamma^{-1}\tilde{x}) = \vartheta_{TN}\big(f(\tilde{x}), c(\gamma)\big).$$

Finalement, on doit traiter l'harmonicité. Une déformation du premier ordre v est dite harmonique si elle est le zéro de l'opérateur de Jacobi, qui a été introduit par Mazet dans [Maz73]. En termes d'un repère orthonormé local $\{E_j\}$, ceci donne:

$$\mathcal{J}(v) = -\sum_{j,k} g^{jk}\Big(\nabla_{E_j}\nabla_{E_k}v + R^N\big(\mathrm{d}f(E_j), v\big)\mathrm{d}f(E_k)\Big) = 0,$$

où ∇ est la connexion pull-back sur f^*TN et R^N est le tenseur de courbure sur N.

Naturellement, les exemples principaux de déformations au premier ordre sont de la forme $v = \frac{\partial f_t}{\partial t}\big|_{t=0}$, où $f_t\colon \tilde{M} \to N$ est une famille lisse d'applications harmoniques, pour un paramètre réel $t \in (-\varepsilon, \varepsilon)$. Alors on vérifie que si chaque f_t est harmonique, alors v l'est, et si chaque f_t est ρ_t-équivariante (pour une famille de représentations ρ_t) alors v est $\rho_t^{(1)}$-équivariante, où $\rho_t^{(1)}$ est déterminé par $c(\gamma) = \frac{\partial \rho_t(\gamma)}{\partial t}\big|_{t=0} \cdot \rho_0(\gamma)^{-1}$. Le résultat principal du chapitre 3 est alors une construction qui donne toutes les déformations $\rho_t^{(1)}$-équivariantes et harmoniques. Pour l'énoncer, soit $\omega \in \mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$ le représentant harmonique de la classe de 1-cohomologie donnée par c (ici, on exploite l'isomorphisme usuel $H^1(\Gamma, \mathfrak{g}) \cong H^1(M, \mathrm{Ad}(\rho_0))$). En tirant arrière ω à $\tilde{M}$, on obtient une 1-forme fermée à valeurs dans $\mathfrak{g}$, qui peut être intégrée à une fonction $F\colon \tilde{M} \to \mathfrak{g}$. De Plus, on peut imposer que F satisfasse la condition d'équivariance:

$$F(\gamma\tilde{x}) = \mathrm{Ad}_{\rho_0(\gamma)}F(\tilde{x}) + c(\gamma).$$

On appelle de telles applications "$\rho_t^{(1)}$-équivariantes et de type harmonique"; on peut démontrer qu'elles existent toujours et en fait elles forment un torseur sur $\mathfrak{h}$, l'algèbre de Lie de H (donc, il s'agit d'un espace affine). Le théorème principal du chapitre 3 s'énonce alors:

Théorème A. *L'application naturelle $\vartheta_{TN}\colon N \times \mathfrak{g} \to TN$ induit une application affine surjective*

$$\left\{ \begin{array}{c} F\colon \tilde{M} \to \mathfrak{g}\ :\ \rho_t^{(1)}\text{-}\textit{équivariante} \\ \textit{de type harmonique} \end{array} \right\} \xrightarrow{\vartheta_{TN}} \left\{ \begin{array}{c} v \in \mathcal{C}^\infty(f^*TN)\ \textit{harmonique} \\ \textit{et } \rho_t\text{-}\textit{équivariante.} \end{array} \right\}$$

Sur des variétés kähleriennes, la construction est fonctorielle.

Dans le chapitre 4 on applique ce résultat pour calculer la variation première de l'énergie d'une famille d'applications harmoniques et équivariantes. La formule (donnée dans la Proposition 4.1.3) est:

$$\frac{\partial E_t}{\partial t}\bigg|_{t=0} = \int_M \big\langle \omega, \beta \big\rangle \mathrm{dVol}_g.$$

Ici, $\frac{\partial E_t}{\partial t}$ est uniquement une notion, qui peut être vue comme la vraie dérivation de n'importe quelle famille lisse f_t qui étend (f, v), qui est supposée être $\rho_t^{(1)}$-équivariante et harmonique. Grâce à ce résultat, on est capable d'enquêter sur les points critiques de l'énergie. Puisque les espaces avec lesquels on travaille ne sont pas lisses, on *définie* un point critique comme un point tel que $\int_M \langle \omega, \beta \rangle = 0$ pour tout $\omega \in \mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$. La lissité générique de l'application $\mathscr{H}$ assure que ceci coïncide avec la définition usuelle dans les points lisses. Alors on démontre:

Théorème B. *Soit $M = X$ une variété compacte kählerienne. Alors les points critiques de la fonctionnelle de l'énergie sur $\mathbb{R}_{\mathrm{B}}(X, G)$ sont exactement les représentations qui proviennent des variations complexes de structures de Hodge.*

La preuve que toute variation complexe de structures de Hodge est un point critique de l'énergie repose sur la remarque que, dans ce cas, β peut être écrit comme $(D'' - D')\gamma$ pour une certaine section γ de $\tilde{X} \times \mathfrak{g}$; ensuite, grâce aux identités de Kähler, on a que ω est à la fois D''- est D'-harmonique. Pour la direction opposée, on enquête sur la déformation spéciale $(\mathcal{E}, t\theta)$ et on démontre que la variation de l'énergie le long de cette direction coïncide avec $\|\omega\|_{L^2}^2$. Ainsi, dans un point critique on doit avoir $\omega = 0$, qui se relie à l'annulation de θ en cohomologie de Dolbeault; ce fait est en fait équivalent à demander que $(\mathcal{E}, \theta)$ soit une variation complexe de structures de Hodge. Cette preuve donne aussi le résultat suivant, qui est en fait une façon différente d'énoncer le théorème:

Corollary. *Si $M = X$ est de Kähler, le gradient de l'énergie défini par rapport à la métrique de Weil-Petersson est (à facteur 2 près) le champ de vecteurs tangent à l'action de $\mathbb{C}^*$.*

Grâce à cela, on a une version "infinitesimale" de l'action $\mathbb{C}^*$ dans le cas riemannian général.

Remarquons que quand $X = \Sigma$ est une surface de Riemann, ce résultat était déjà dans [Hit87]: comme l'énergie est une application moment pour l'action du cercle, un point est critique si et seulement s'il est fixé; mais être fixé par S^1 ou par $\mathbb{C}^*$ est en fait la même chose. Ceci nous permet de déduire aisément Théorème B dans le cas où X est projectif et la classe de Kähler est intégrale est très ample; notre théorème n'est donc vraiment nouveau que dans le cas Kähler général (ou bien projectif mais avec une classe de Kähler qui n'est pas intègre).

Analyse du second ordre

On procède, pour l'étude du second ordre d'une application harmonique, sur les mêmes procédés; par contre, les définitions deviennent moins naturelles et on trouve des obstructions à l'existence des déformations au second ordre.

Une déformation au second ordre d'une application $f\colon \tilde{M} \to N$ peut être définie soit comme une section de f^*J^2N, le fibré tiré arrière du fibré des 2-jets, soit comme un couple (v, w) de sections de f^*TN. Les deux définitions sont liées par la connexion canonique, qui donne un isomorphisme entre les deux fibrés. On a choisi la deuxième approche; pour fixer les idées, pour une famille lisse d'applications $f_t\colon \tilde{M} \to N$, la section w est définie par

$$w = \frac{D}{\partial t}\frac{\partial f_t}{\partial t}\Big|_{t=0},$$

où $\frac{D}{\partial t} = \nabla_{\frac{\partial}{\partial t}}$ est la dérivée covariante le long de $\frac{\partial f_t}{\partial t}\big|_{t=0}$. De la même façon que dans le cas du premier ordre, on peut commencer par une famille d'applications ρ_t-équivariantes (resp. harmoniques) et déduire des formules pour $\rho_t^{(2)}$-équivariantes (resp. harmoniques) déformations du second ordre. Des définitions précises se trouvent dans la section 5.1; ici, on se limite à remarquer que la condition pour l'harmonicité s'exprime encore en termes de l'opérateur de Jacobi $\mathcal{J}$ et du tenseur de courbure riemannienne R^N, et qu'à nouveau une déformation du second ordre $\rho_t^{(2)}$ de $\rho_t^{(1)}\colon \Gamma \to TG$ est la même chose d'une application $k\colon \Gamma \to \mathfrak{g}$ qui rend le couple (c, k) un 1-cocycle pour l'action adjointe $\mathrm{Ad}(\rho_t^{(1)})$ de Γ sur $\mathfrak{g} \times \mathfrak{g} \cong \mathfrak{g} \otimes \mathbb{R}[t]/(t^2)$.

Ensuite, on procède à définir des fonctions $\rho_t^{(2)}$-équivariantes et de type harmonique $(F, F_2)\colon \tilde{M} \to \mathfrak{g} \times \mathfrak{g}$. La définition d'équivariance est dans le même esprit que pour le premier ordre: on demande que

$$(F, F_2)(\gamma\tilde{x}) = \rho_t^{(1)}(\gamma) \cdot \big(F(\tilde{x}), F_2(\tilde{x})\big) + (c(\gamma), k(\gamma)).$$

Par contre, l'harmonicité est plus compliquée: la 1-forme $\mathrm{d}F_2$ n'est même pas une section de $\mathrm{Ad}(\rho_0)$, donc on la modifie en définissant la 1-forme ψ par:

$$\psi = \mathrm{d}F_2 + [\omega, F].$$

Cette 1-forme est à valeurs dans $\mathrm{Ad}(\rho_0)$, mais ni fermée ni co-fermée. On dit que (F, F_2) est de type harmonique si ψ satisfait les équations suivantes (en termes d'un repère local orthonormé $\{E_j\}$):

$$\mathrm{d}\psi = -[\omega, \omega]; \quad \mathrm{d}^*\psi \overset{\mathrm{loc}}{=} \sum_j \big[\sigma(\omega(E_j)), \omega(E_j)\big], \tag{5.10}$$

où σ est l'involution du système local harmonique polarisé sur $\tilde{M} \times \mathfrak{g}$. On peut alors définir une application $\vartheta_{J^2N} \colon N \times \mathfrak{g} \times \mathfrak{g} \to TN \times_N TN$ par:

$$\vartheta_{J^2N}(n, \xi, \mu) = \Big(\vartheta_{TN}(n, \xi), \vartheta_{TN}\Big(n, \mu + \big[\xi^{[\mathfrak{k}]}, \xi^{[\mathfrak{p}]}\big]\Big)\Big).$$

Ici, $[\mathfrak{k}] = [\mathfrak{k}]_n$ (resp. $[\mathfrak{p}]$) denote la projection sur l'espace propre de valeur propre 1 (resp. -1) de σ. En particulier, cette application n'est plus linéaire, mais on peut quand même démontrer qu'il envoie des fonctions $\rho_t^{(2)}$-équivariantes et de type harmonique (F, F_2) dans des déformations harmoniques et $\rho_t^{(2)}$-équivariantes.

Par conséquent, on a à nouveau une façon de construire des déformations harmoniques et $\rho_t^{(2)}$-équivariantes du second ordre. Par contre, contrairement au cas du premier ordre, il y a des obstructions à l'existence de ces objets. On collecte les résultats principaux dans le théorème suivant:

Théorème C. *Soit $\rho_t^{(2)} \colon \Gamma \to J^2G$ une déformation au second ordre de $\rho_t^{(1)} = (\rho_0, c)$ et $f \colon \tilde{M} \to N$ une application harmonique et ρ_0-équivariante. L'existence d'une 1-forme ψ qui satisfait* (5.10) *implique que chaque déformation du premier ordre (f, v) qui est harmonique et $\rho_t^{(1)}$-équivariante s'étend à une déformation du second ordre (f, v, w) harmonique et $\rho_t^{(2)}$-équivariante. De plus:*

- *Si G est complexe, il y a une réciproque: une telle ψ existe si, et seulement si, on peut trouver deux déformations harmoniques du second ordre (f, v, w) et $(f, \tilde{v}, \tilde{w})$, équivariantes l'une pour (ρ_0, c) et l'autre pour (ρ_0, ic) (les seconds ordres des représentations ne jouent aucun rôle).*

- *La 1-forme ψ existe si, et seulement si, ω est un point critique de la norme L^2 dans son orbite sous l'action adjointe de $H = Z_G(\mathrm{Image}(\rho_0))$ sur $\mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$. Par conséquent, si G est complexe et si (ρ_0, c) est donné, il existe une métrique f pour laquelle ψ existe si, et seulement si, ω est un point polystable pour cette action.*

- *Une condition suffisante pour l'existence de ψ pour toute métrique f est que $H^1(M, \mathrm{Ad}(\rho_t^{(1)}))$ soit un $\mathbb{R}[t]/(t^2)$-module plat. Si G est complexe, cette condition est aussi nécessaire.*

On donne des exemples de $\rho_t^{(1)} = (\rho_0, c)$ pour lesquels il n'y a aucune métrique f admettant une déformation au second ordre (f, v, w) à la fois harmonique est $\rho_t^{(2)}$-équivariant (encore une fois, pour n'importe quel $\rho_t^{(2)}$

qui étend $\rho_t^{(1)}$). Par contre, à ce moment on ne connaît aucun exemple où (ρ_0, c) admet une telle métrique mais (ρ_0, ic) ne l'admet pas, c'est-à-dire, un exemple d'une f déformable telle que ψ n'existe pas.

Sous l'assomption de l'existence de ψ, on peut donner une formule pour la variation seconde de l'énergie, qui ressemble celle pour les variations premières:

$$\frac{\partial^2 E_t}{\partial t^2}\bigg|_{t=0} = \int_M \Big(\big\langle \psi, \beta \big\rangle + \|\omega^{[\mathfrak{p}]}\|^2\Big) \mathrm{dVol}_g.$$

Avec ce résultat à disposition, on peut démontrer le théorème suivant:

Théorème D. *Si G est un groupe complexe, l'énergie est une fonction strictement pluri sous-harmonique sur les points lisses de l'espace des modules. Par conséquent, il donne une fonction pluri sous-harmonique sur sa normalisation.*

En fait, il s'avère que l'énergie est un potentiel de Kähler pour la structure complexe de Betti avec la métrique de Weil-Petersson.

Finalement, dans le cas kählerien, on arrive à étudier la positivité de la matrice hessienne de l'énergie dans les points critiques, en termes des valeurs propres du générateur local de l'action de S^1, dans le même esprit que [Hit92]. Pour souligner le parallélisme, on introduit la notation suivante: si α est une 1-forme à valeurs dans $\mathfrak{g}$, notons $\alpha^{[\mathfrak{k}]}$ (resp. $\alpha^{[\mathfrak{p}]}$) la 1-forme telle que pour chaque champ tangent (réel) $\chi \in \Xi(X)$, les exposants $[\mathfrak{k}]$ et $[\mathfrak{p}]$ dénotent les projections sur les sous-espaces correspondants:

$$\alpha^{[\mathfrak{k}]}(\chi) = \alpha(\chi)^{[\mathfrak{k}]} \in [\mathfrak{k}] = \mathrm{Ker}(\sigma - \mathrm{Id}), \quad \alpha^{[\mathfrak{p}]}(\chi) = \alpha(\chi)^{[\mathfrak{p}]} \in [\mathfrak{p}] = \mathrm{Ker}(\sigma + \mathrm{Id}).$$

De plus, soit α' (resp. α'') la partie holomorphe (resp. anti-holomorphe) de α. Définissons $\dot{A} = (\omega^{[\mathfrak{k}]})''$ et $\dot{\Phi} = (\omega^{[\mathfrak{p}]})'$ (voir [Hit87] pour une interprétation plus directe de ces symboles dans le cas d'une surface de Riemann $X = \Sigma$).

Théorème E. *Soit G un groupe complexe, $\rho_0 \colon \Gamma \to G$ induite par une variation de structures de Hodge complexes et $f \colon \tilde{X} \to N$ l'application harmonique induite par l'application des périodes. Alors, la variation du second ordre de l'énergie le long de $\omega \in \mathcal{H}^1(X, \mathrm{Ad}(\rho_0))$ peut s'écrire comme*

$$\frac{\partial^2 E_t}{\partial t^2}\bigg|_{t=0} = 2\int_X \bigg(\sum_p -p\big\|\dot{A}^{-p,p}\big\|^2 + (1-p)\big\|\dot{\Phi}^{-p,p}\big\|^2\bigg) \mathrm{dVol}_g,$$

où $\xi = \sum_p \xi^{-p,p}$, pour $\xi \in \mathfrak{g}$, dénotes la décomposition de Hodge de $\mathfrak{g}$ induite par la variation complexe de structure de Hodge. Dans le cas particulier où ω est à valeurs dans l'algèbre de Lie de l'adhérence de Zariski de $\mathrm{Image}(\rho_0)$, *cette expression peut se récrire en termes de la structure de Hodge-Deligne* $\omega = \sum_{P+Q=1} \omega^{(P,Q)}$*:*

$$\frac{\partial^2 E_t}{\partial t^2}\Big|_{t=0} = \int_X \Bigg(\sum_{P\equiv 0} P \big\|\omega^{(P,Q)}\big\|^2 + \sum_{P\equiv 1} Q \big\|\omega^{(P,Q)}\big\|^2 \Bigg) \mathrm{dVol}_g.$$

Corollaire. *Si la représentation* ρ_0 *est induite par une variation complexe de structures de Hodge dont le domaine des périodes est un espace symétrique du type hermitien, alors la matrice hessienne de l'énergie en* ρ_0 *est semi-définie positive le long de toute déformations* $\rho_t \colon \Gamma \to G_0$*, où* G_0 *dénote l'adhérence de Zariski réelle de* $\mathrm{Image}(\rho_0)$*.*

Une prépublication issue des résultats de cette thèse est disponible en ligne [Spi13].

Structure de la thèse

Le chapitre 1 est dédié au contexte et aux résultats techniques sur les systèmes locaux harmoniques polarisés. La définition est donnée dans la section 1.2 et l'exemple principal est traité dans la section 1.3. Après avoir brièvement discuté du fibré tangent d'un espace riemannien symétrique du type non-compacte, les résultats principaux sur les systèmes locaux harmoniques polarisés sont donnés dans les sections 1.5 et 1.6. Dans les sections 1.7 et 1.8 on discute les cas spéciales $M = S^1$ et $M = X$ kahlerien. Finalement, la notion de variation complexe de structures de Hodge est révisée dans la section 1.9.

Dans le chapitre 2 on étend le théorème de Corlette en construisant une “application harmonique tordue universelle”. On démontre qu'elle est continue dans la section 2.2; avec un peu plus de travail, on peut alors démontrer que l'énergie est continue sur tout $\mathbb{R}_{\mathrm{B}}(M, G)$.

Le chapitre 3 est consacré à l'étude des déformations au premier ordre des applications harmoniques, avec le but de démontrer le théorème A. Les déformations équivariantes du premier ordre sont décrites dans la section 3.2 et celles harmoniques dans la section 3.3. Dans la section 3.4 on montre la procédé constructive du théorème A et on démontre qu'en fait il donne toute déformation harmonique et équivariante dans la section 3.5. Finalement, on donne un énoncé précis pour le théorème A dans la section 3.6, et dans 3.7 on analyse les résultats quand $G = \mathrm{GL}(1, \mathbb{C})$ (dans ce cas, la cohomolgie non-abelienne se réduit à celle habituelle).

Les résultats ainsi obtenus sont appliqués dans le chapitre 4 à l'étude de la variation première de la fonctionnelle de l'énergie, qui a comme premier objectif de démontrer la formule en termes du produit scalaire entre ω et β

cité ci-dessus. Ceci nous permet de démontrer que les variations complexes de structures de Hodge sont les points critiques de l'énergie: ceci est fait dans la section 4.2. Dans 4.3 on démontre l'autre implication, analysant la variation de l'énergie par rapport à l'action de $\mathbb{C}^*$.

L'analyse du second ordre est effectuée dans le chapitre 5, qui est consacré à la preuve du théorème C. On commence en donnant les définitions et les liens entre eux dans les sections 5.1 et 5.2. Ensuite, on introduit l'action de H sur $\mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$ et en termes de cette action on construit des applications $\rho_t^{(2)}$-équivariantes et de type harmonique (F, F_2). Ceci est fait dans les sections 5.3 et 5.4. Le lien entre l'existence de (F, F_2) et celle d'une déformation (v, w) de f est analysé dans la section 5.5. La section 5.6 est dédiée à l'exemple d'une représentation $\rho_t^{(1)}$ pour laquelle il n'existe aucune métrique déformable au second ordre, et dans la section 5.8 on démontre que quand on a une famille lisse d'applications f_t, pour $t \in (-\varepsilon, \varepsilon)$, les définitions données coïncides avec les définitions naturelles. Remarquons que ce qu'on a noté comme Théorème C dans cette introduction est scindé en Proposition 5.4.6, Proposition 5.5.5, Théorème 5.7.1 et Proposition 5.7.2.

Finalement, dans le chapitre 6 on applique les résultats du chapitre précédent pour obtenir la formule pour la variation seconde de l'énergie en termes de ω et ψ (sous l'assomption que celle-ci existe). Ceci nous permet de démontrer d'un côté la pluri sous-harmonicité de l'énergie du théorème D, ce qui est fait dans la section 6.2. De l'autre côté, on obtient la formule du théorème E dans la section 6.3.

Chapter 1

Riemannian symmetric spaces and Polarized harmonic local systems

Introduction au chapitre

Dans ce chapitre on introduit la notion de systèmes locaux harmoniques polarisés sur une variété riemannienne M qui représentent l'outil technique principal qu'on va exploiter dans la suite de la thèse. Il s'agit d'étudier des fibrés harmoniques $(\mathcal{V}, D, h)$ (c'est-à-dire, des fibrés $\mathcal{V}$ avec une connexion plate D et une métrique h qui induit une application harmonique $\tilde{M} \to \mathrm{GL}(r, \mathbb{C})/O(r)$, r étant le rang de $\mathcal{V}$), munis de plus d'une involution $\sigma\colon \mathcal{V} \to \mathcal{V}$ telle que, si on définit $S(v, w) = h(v, \sigma(w))$, alors S donne une forme quadratique parallèle, symétrique ou anti-symétrique. Une notion analogue existe pour des fibrés complexes (dans ce cas, on demande que S et h soient hermitiens). L'exemple principal de système local harmonique polarisé est défini comme suit: soit G un groupe algébrique réductif, $N = G/K$ l'espace symétrique riemannien associé et $(\mathcal{V}_{\mathrm{ad}}, D) = (N \times \mathfrak{g}, \mathrm{d})$, $\mathfrak{g}$ étant l'algèbre de Lie de G. En notant par σ_0 l'involution de Cartan associée au compact maximal K, on définit $\sigma_{\mathrm{ad}}(v)$ en dessus d'un point $n \in N$ par $\mathrm{Ad}_n(\sigma_0(\mathrm{Ad}_{n^{-1}} v))$. En prenant une forme symétrique induite par la forme de Killing sur la partie semi-simple de $\mathfrak{g}$ on obtient un système local harmonique polarisé (en fait, la métrique hermitienne h ainsi définie est même totalement géodésique), qu'on appellera système adjoint. On peut voir que, donné n'importe quel autre système local polarisé harmonique $\mathcal{V}$ sur une variété riemannienne M, celui induit sur $\mathrm{End}(\mathcal{V})$ est essentiellement le pullback par l'application harmonique du système adjoint.

Tous les systèmes locaux harmoniques polarisés sont munis d'une connexion canonique $\mathrm{d}^{\mathrm{can}}$ donnée par la partie métrique de D. En écrivant $\mathrm{d} = \mathrm{d}^{\mathrm{can}} + \beta$, on obtient une 1-forme β à valeurs dans les endomorphismes auto-adjoints de $\mathcal{V}$. Grâce à la description usuelle du fibré tangent d'un espace symétrique, en notant par $N \times \mathfrak{g} = [\mathfrak{k}] \oplus [\mathfrak{p}]$ la décomposition en espaces propres de σ, on a alors que le pull-back de β à $\tilde{M}$ donne une 1-forme $\tilde{\beta} \in \mathcal{A}^1_{\tilde{M}}([\mathfrak{p}])$. On démontre dans la Proposition 1.5.7 que $\tilde{\beta}$ correspond à $\mathrm{d}f$ sous l'identification canonique $\vartheta_{TN} \colon [\mathfrak{p}] \xrightarrow{\sim} TN$ et que $\mathrm{d}^{\mathrm{can}}$ commute à σ.

Les résultats sur les systèmes locaux harmoniques polarisés les plus importants pour la suite sont démontrés dans la Section 1.6, sous l'hypothèse de compacité pour M. Notamment, on donne une formule pour le laplacien $\Delta = \mathrm{d}^*\mathrm{d}$ dans le Lemme 1.6.6, qui entraine en particulier la commutation entre Δ et σ. Une autre conséquence importante est donnée dans le Corollaire 1.6.9, qui implique que l'espace vectoriel des sections globales plates admet une décomposition plate induite par σ.

Deux cas particuliers sont traités dans les dernières trois sections. Premièrement, quand $M = S^1$, les applications harmoniques définies sur $\tilde{M} = \mathbb{R}$ sont des géodésiques; on révise comment les résultats d'existence des applications harmoniques tordues de Corlette [Cor88] peuvent se relire dans ce contexte en termes des isométries semi-simples, l'énergie étant le carré de la distance de translation. Deuxièmement, on considère le cas $M = X$ d'une variété kählerienne. Cela étant le cas motivant de l'analyse, on explique comment les variations polarisées de structure de Hodge (réelles ou complexe) sont des prototypes de systèmes locaux polarisés harmoniques sur ces variétés. Dans ce cas, les systèmes locaux harmoniques polarisés possèdent aussi des meilleures propriétés de fonctorialité par rapport au cas général; on donne le lien avec la théorie des fibrés de Higgs, fournissant des interprétations des objets introduits dans le chapitre. En particulier, on donne des preuves de la cloture de la 2-forme $\mathrm{trace}(\theta \wedge \theta^*)$, qui est aussi une conséquence d'un Théorème de Mok [Mok92], et du fait que, en conséquence des identités de Kähler, le pull-back d'une 1-forme harmonique à valeurs dans un fibré harmonique est encore harmonique.

1.1 Definitions and notations

The purpose of this section is to fix notations and list some standard results. For the ease of reference, concerning Riemannian geometry and harmonic mappings we will try to keep the same notations and conventions as [EL83].

Let (M, g) be a connected, orientable Riemannian manifold, and let $E \to M$ be a vector bundle on M. We denote by $\mathcal{A}^p(E) = \Lambda^p(T^*M) \otimes E$ the real vector space of E-valued smooth p-forms, and by $\mathcal{C}^\infty(E) = \mathcal{A}^0(E)$ the space of smooth sections.

Denote by TM the tangent bundle of M and by $\mathcal{C}^\infty(M)$ the space of smooth real functions on M, so that $\mathcal{C}^\infty(TM)$ is a $\mathcal{C}^\infty(M)$-module. Given a connection ∇ on E, we still denote by ∇ its extension to E-valued forms. Further, we define the associated exterior differential operator $\mathrm{d}\colon \mathcal{A}^p(E) \to \mathcal{A}^{p+1}(E)$ as the anti-symmetrization of ∇ (cfr. [EL83], (1.15) and (1.16)).

For any connection ∇ on $E \to M$ we define its curvature $R\colon \Lambda^2\mathcal{C}^\infty(TM) \otimes \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ by

$$R(X,Y)\sigma = -\nabla_X\nabla_Y\sigma + \nabla_Y\nabla_X\sigma + \nabla_{[X,Y]}\sigma,$$

so that for every $\phi \in \mathcal{A}^p(E)$,

$$\mathrm{d}^2\phi = -R \wedge \phi.$$

Given a connection ∇ and a metric $\langle\cdot,\cdot\rangle$ on E, and letting d be as above, we define the codifferential operator $\mathrm{d}^*\colon \mathcal{A}^p(E) \to \mathcal{A}^{p-1}(E)$ associated to $\langle\cdot,\cdot\rangle$[1] by:

$$\int_M \langle \mathrm{d}\phi, \psi\rangle \mathrm{dVol}_g = \int_M \langle \phi, \mathrm{d}^*\psi\rangle \mathrm{dVol}_g,$$

where $\phi \in \mathcal{A}^{p-1}(E)$, $\psi \in \mathcal{A}^p(E)$, with ϕ compactly supported, and dVol_g denotes the volume unit for the metric g.

Given a metric and a connection, we define the associated Laplacian as $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*\colon \mathcal{A}^p(E) \to \mathcal{A}^p(E)$. We say that a p-form ϕ is *harmonic* (with respect to the given metric and connection) if $\Delta\phi = 0$.

A direct computation in [EL83], (1.20) proves that if ∇ is metric with respect to $\langle\cdot,\cdot\rangle$ and M is compact, then for every 1-form $\alpha \in \mathcal{A}^1(E)$ and $\{E_i\}$ a local frame of TM,

$$\mathrm{d}^*\alpha = -\mathrm{trace}\nabla\alpha = -\sum_{i,j} g^{ij}\nabla_{E_j}\alpha(E_i). \tag{1.1}$$

Let M, N be Riemannian manifolds, with M compact, and let $\tilde{\pi}\colon \tilde{M} \to M$ be the universal cover of M. Let $\rho\colon \Gamma = \pi_1(M, x_0) \to \mathrm{Isom}(N)$ be a

[1]Or, rather, to the metric induced by $\langle\cdot,\cdot\rangle$ and g, that is, for p-forms ϕ and ϕ', E_i a local frame of TM and g^{ij} the inverse matrix of $g(E_i, E_j)$,

$$\langle\phi,\phi'\rangle = \sum_{\substack{i_1<\ldots<i_p\\ j_1<\ldots<j_p}} g^{i_1j_1}\cdots g^{i_pj_p}\langle\phi(E_{i_1},\ldots,E_{i_p}), \phi'(E_{j_1},\ldots,E_{j_p})\rangle$$

representation. A smooth mapping $f\colon \tilde{M} \to N$ is called *harmonic* if it is an extremal point of the energy

$$E(f') = \int_{\tilde{M}/\Gamma} \left|\mathrm{d}f'\right|^2,$$

amongst all ρ-equivariant maps $f'\colon \tilde{M} \to N$. Here the integral is equivalently taken over any fundamental domain for the action of Γ on $\tilde{M}$, and the norm involves both the metric on M and that on N. A function f is harmonic if and only if its tension field vanishes, that is, it satisfies the Euler-Lagrange equation:

$$\tau(f) = \operatorname{trace}\bigl(\nabla \mathrm{d}f\bigr) \overset{\text{loc}}{=} \sum_i \nabla_{E_i} \mathrm{d}f(E_i) = 0. \tag{1.2}$$

Here the connection is induced by the Levi-Civita connection on N, and the trace is in terms of the Riemannian metric on M, that is, E_i is a local orthonormal frame for $\tilde{M}$.

Recall the following fundamental result for the existence of harmonic maps:

Theorem 1.1.1 ([Cor88]). *Let M be a compact manifold, and $\Gamma = \pi_1(M, x_0)$ its fundamental group. Further let G be a real reductive algebraic group, $\rho\colon \Gamma \to G$ a representation, G_0 the real Zariski closure of its image and $K_0 < K$ maximal compact subgroups of G_0, G, respectively. There exists a ρ-equivariant harmonic map $f\colon \tilde{M} \to G_0/K_0 \subset G/K$ if, and only if, G_0 is reductive. Furthermore, this map is unique up to isometry of the ambient space commuting with G_0 (i.e. the ρ-equivariant harmonic map $f\colon \tilde{M} \to G/K$ is unique up to multiplication by an element of the centralizer $H = Z_G(G_0)$).*

1.2 Complex and real polarized harmonic local systems

We are interested in the study of certain local systems on M enjoying special properties. Simpson [Sim92] introduced the following notion of *harmonic bundles*:

Definition 1.2.1. A harmonic bundle is a flat complex bundle $(\mathcal{V}, D)$ on M together with a harmonic metric h, that is, such that the map it defines:

$$f_h\colon \tilde{M} \to \mathrm{GL}(n, \mathbb{C})/U(n), \tag{1.3}$$

is harmonic.

We recall briefly how the map $f = f_h$ is constructed from h. Fix a base point $x_0 \in M$ and an isomorphism $\mathcal{V}_{x_0} \cong \mathbb{C}^n$, so that $\tilde{\pi}^*\mathcal{V} \cong \tilde{M} \times \mathbb{C}^n$. Then h gives a positive definite $n \times n$ hermitian matrix at every point $\tilde{x} \in \tilde{M}$, say $A(\tilde{x})$. Since the symmetric space $\mathrm{GL}(n)/U(n)$ classifies such matrices, we get a function f as stated, which is such that $f(\tilde{x}) \cdot f(\tilde{x})^* = A(\tilde{x})^{-1}$. Indeed, for every v, w sections of $\tilde{\pi}^*\mathcal{V}$, we have

$$h(v, w) = \left\langle v(\tilde{x}), A(\tilde{x})w(\tilde{x})\right\rangle_{\mathbb{C}^n} = \left\langle f(\tilde{x})^{-1}v(\tilde{x}), f(\tilde{x})^{-1}w(\tilde{x})\right\rangle_{\mathbb{C}^n} \qquad (1.4)$$

(here, $f(\tilde{x})^{-1}$ is defined only after taking a lifting $\tilde{M} \to \mathrm{GL}(n, \mathbb{C})$ of f, but the result is independent of the chosen lift). Remark that the choice of a base point $x_0 \in M$ and a trivialization $\mathcal{V}_{\tilde{x}_0} \xrightarrow{\sim} \mathbb{C}^n$ also gives a monodromy representation $\rho\colon \pi_1(M, x_0) \to \mathrm{GL}(n, \mathbb{C})$, and the f thus defined is ρ-equivariant.

Let us introduce a class of harmonic bundles having slightly better features than general ones.

Definition 1.2.2. A triple $(\mathbb{V}, \sigma, S)$ will be called a complex polarized harmonic local system ($\mathbb{C}$-phls for short) if it consists of:

1. A local system $\mathbb{V}$ of $\mathbb{C}$-vector spaces;

2. A flat $\mathbb{C}$-vector bundle $(\mathcal{V}, D)$ such that $\mathbb{V} = \ker(D)$;

3. A hermitian, non-degenerate form S on $\mathcal{V}$, which is flat, that is, $D(S) = 0$;

4. A $\mathbb{C}$-linear involution $\sigma\colon \mathcal{V} \to \mathcal{V}$ such that $\mathcal{V}^+ = \ker(\sigma - \mathrm{Id})$ and $\mathcal{V}^- = \ker(\sigma + \mathrm{Id})$ are orthogonal with respect to S, which is positive definite on the former and negative definite on the latter;

5. The induced positive definite metric, $\langle v, w\rangle = h(v, w) = S(v, \sigma(w))$, is required to be *harmonic.*

In particular, of course, $(\mathcal{V}, D, h)$ is a harmonic bundle.

Lemma 1.2.3. *Let $(\mathbb{V}, \sigma, S)$ be a complex phls on M. Fix a point $x_0 \in M$, and an isomorphism $\mathbb{V}_{x_0} \xrightarrow{\eta} \mathbb{C}^n$, so that we have a monodromy representation $\rho\colon \Gamma = \pi_1(M, x_0) \to \mathrm{GL}(n, \mathbb{C})$. Let G_0 be the real Zariski closure of the image of ρ. Then G_0 is contained in $U(S \circ \eta^{-1}) \cong U(r^+, r^-)$ where $r^{\pm} = \mathrm{rk}\mathcal{V}^{\pm}$, and S_{x_0} is G_0-invariant, that is, for $g \in G_0$ and v, w in $\mathcal{V}_{x_0}$,*

$$S(g \cdot v, g \cdot w) = S(v, w).$$

Proof. The first assertion follows from the second one, since the unitary group for S is isomorphic to $U(r^+, r^-)$. The second statement follows from flatness of S, which implies that S is $\rho(\Gamma)$-invariant. □

Corollary 1.2.4. *Let $(\mathbb{V}, \sigma, S)$ be a $\mathbb{C}$-phls on M, and denote by G_0 the monodromy group and by $\mathfrak{g}_0$ its Lie algebra. Then the hermitian form S is $\tilde{M} \times_\Gamma \mathfrak{g}_0$ invariant, where Γ acts on G_0 by $\mathrm{Ad}(\rho)$, meaning that for every $A_0 \in \mathcal{C}^\infty(\tilde{M} \times_\Gamma \mathfrak{g}_0)$,*

$$S(A_0(v), w) + S(v, \overline{A_0}(w)) = 0.$$

Notation 1.2.5. From now on, we will denote by G_0 the real Zariski closure of the image of the monodromy, K_0 a maximal compact subgroup, and $N_0 = G_0/K_0$. Unless otherwise stated, we will always assume that G_0 be reductive (the motivation for this lies in Corlette's theorem 1.1.1). We will denote by f_0 a harmonic metric taking values in the totally geodesic subset N_0 of $\mathrm{GL}(n, \mathbb{C})/U(n)$ (cfr. [Hel78], chapter IV, theorem 7.2), and $H = Z_{\mathrm{GL}(n,\mathbb{C})}(G_0)$ the centralizer of G_0 (or, equivalently, of $\mathrm{Image}(\rho)$). Later on, we will work with intermediate reductive groups $G_0 \subseteq G \subseteq \mathrm{GL}(n, \mathbb{C})$, in which case we will denote by $H = Z_G(G_0)$.

Let us explain the reasons behind this. Corlette's theorem 1.1.1 implies that, for f to exist, the group G_0 must be reductive. Under this hypothesis, it also grants the existence of a harmonic metric $f_0 \colon \tilde{M} \to G_0/K_0$. Furthermore, all harmonic metrics with values in some (possibly bigger) symmetric space $N = G/K$ only differ by an isometry of this space, hence by multiplication by a fixed element g of G. Equivariance implies that this element must commute with $\mathrm{Image}(\rho)$, that is, $g \in H$. Thus, given any harmonic metric $f \colon \tilde{M} \to N$, we can always find a f_0 as in the notations above and an $h \in H$ such that $f = h \cdot f_0$.

Definition 1.2.6. A real (even or odd) polarized harmonic local system is a triple $(\mathbb{V}_\mathbb{R}, \sigma, Q)$, where:

1. The local system $\mathbb{V}_\mathbb{R}$ is of real vector spaces;
2. There is a flat $\mathbb{R}$-vector bundle $(\mathcal{V}_\mathbb{R}, D)$ such that $\mathbb{V}_\mathbb{R} = \ker(D)$;
3. The flat, non-degenerate form Q is symmetric in the even case and symplectic in the odd case;
4. There is an $\mathbb{R}$-linear involution $\sigma \colon \mathcal{V}_\mathbb{R} \to \mathcal{V}_\mathbb{R}$ such that, again denoting the eigenspaces by $\mathcal{V}^+$ and $\mathcal{V}^-$, they are orthogonal, and, in the even

case, Q is positive definite on $\mathcal{V}^+$ and negative definite on $\mathcal{V}^-$. In the odd case, defining,

$$S(v,w) = iQ_{\mathbb{C}}(v,\bar{w}), \quad \forall v,w \text{ sections of } \mathcal{V}_{\mathbb{C}} = \mathcal{V}_{\mathbb{R}} \otimes \mathbb{C},$$

the complexification of the involution $\sigma\colon \mathcal{V}_{\mathbb{C}} \to \mathcal{V}_C$ is required to satisfy the same properties as in the complex setting of definition 1.2.2.

5. In both cases, the positive definite metric (which is symmetric on $\mathcal{V}_{\mathbb{R}}$ in the even case and hermitian on $\mathcal{V}_{\mathbb{C}}$ in the odd case) is required to be harmonic.

The same notational remarks after definitions of complex phls apply here, too. Letting again G_0 denote the real Zariski closure of the monodromy of D and H its centralizer, then there exists $f_0\colon \tilde{M} \to G_0/K_0$, such that $f = h \cdot f_0$ for some $h \in H$ and

$$\begin{cases} G_0/K_0 \subseteq O(r^+,r^-)/(O(r^+) \times O(r^-)) \subset \mathrm{GL}(n,\mathbb{R})/O(n) & \text{in the even case} \\ G_0/K_0 \subseteq \mathrm{Sp}(2n,\mathbb{R})/U(n) \subset \mathrm{GL}(n,\mathbb{C})/U(n) & \text{in the odd case.} \end{cases}$$

Lemma 1.2.7. *Let* $(\mathbb{V}_{\mathbb{R}}, \sigma_{\mathbb{R}}, Q_{\mathbb{R}})$ *be a* $\mathbb{R}$*-phls. Then we obtain a* $\mathbb{C}$*-phls* $(\mathbb{V}, \sigma, S)$ *by defining, in the even case:*

$$\mathbb{V} = \mathbb{V}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}; \quad \sigma(v) = \sigma_{\mathbb{C}}(v); \quad S(v,w) = Q_{\mathbb{C}}(v,\bar{w}),$$

for all v,w *in* $\mathcal{V}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$*, where* $\sigma_{\mathbb{C}}$ *and* $Q_{\mathbb{C}}$ *denote the* $\mathbb{C}$*-multilinear extensions of* $\sigma_{\mathbb{R}}$ *and* $Q_{\mathbb{R}}$*, respectively.*

Proof. For odd $\mathbb{R}$-phls, the statement is part of the definition. In the even case, the verification of the first four properties of the definition are immediate. For harmonicity, we only have to remark that if $f\colon \tilde{M} \to \mathrm{GL}(n,\mathbb{R})/O(n)$ is harmonic, then

$$i \circ f\colon \tilde{M} \to \mathrm{GL}(n,\mathbb{R})/O(n) \to \mathrm{GL}(n,\mathbb{C})/U(n)$$

is, as well, because the map $i\colon \mathrm{GL}(n,\mathbb{R})/O(n) \to \mathrm{GL}(n,\mathbb{C})/U(n)$ is totally geodesic. □

Lemma 1.2.8. *There are natural definitions of tensor products and duals of (real or complex) phls; in the real case, the tensor product is additive on parity, and dualizing preserves it. In particular, if* $(\mathcal{V}, \sigma, S)$ *is a phls on a flat bundle* $\mathcal{V}$*, the bundle* $\mathrm{End}(\mathcal{V})$ *supports a structure of phls (which is always even in the real case).*

Proof. Everything follows at once from the definitions. For example, if $(\mathbb{V}, \sigma_V, S_{\mathbb{V}})$ and $(\mathbb{W}, \sigma_{\mathbb{W}}, S_{\mathbb{W}})$ are two phls, then the non-degenerate form and the involution on the tensor product are defined on pure products as

$$S(v \otimes w, v' \otimes w') = S_{\mathbb{V}}(v, v')S_{\mathbb{W}}(w, w'), \quad \sigma(v \otimes w) = \sigma_{\mathbb{V}}(v) \otimes \sigma_{\mathbb{W}}(w).$$

Again, for example in the real case, harmonicity of the obtained metric follows from the "tensor map"

$$\mathrm{GL}(n, \mathbb{R})/O(n) \times \mathrm{GL}(m, \mathbb{R})/O(m) \to \mathrm{GL}(nm, \mathbb{R})/O(nm)$$

being totally geodesic. This map associates to the symmetric, positive definite matrices $A \in M_n(\mathbb{R})$, $B \in M_m(\mathbb{R})$ the symmetric and positive definite matrix $A \otimes B$ such that, if $\{e_i\}$ and $\{f_j\}$ are the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively,

$$(e_i \otimes f_j)(A \otimes B)(e_k \otimes f_h) = A_{ik}B_{jh}.$$

For the dual, we only remark that the map $\mathrm{GL}(n, \mathbb{R})/O(n) \to \mathrm{GL}(n, \mathbb{R})/O(n)$ associating to a metric f the metric on the dual bundle $\mathcal{V}^\vee$ is the one sending the symmetric matrix $A(\tilde{x}) = (f(\tilde{x}) \cdot f(\tilde{x})^t)^{-1}$ to its inverse $A(\tilde{x})^{-1}$. This is because, if we write $\mathcal{V} = \widetilde{M} \times_\Gamma \mathbb{R}^n$, with Γ acting on $\mathbb{R}^n$ by $\rho(\tilde{x})$, the dual structure is obtained by the action $(\rho(\tilde{x})^t)^{-1}$. The complex hermitian case is completely analogous. □

1.3 The adjoint polarized harmonic linear system

Let $\mathbb{G}$ be a reductive connected real algebraic group, and denote by $G = \mathbb{G}(\mathbb{R})$. Let $K < G$ be a maximal compact subgroup and denote by $N = G/K$ the associated Riemannian symmetric space (which is diffeomorphic to $\mathbb{R}^n$, see [Hel78], Chapter VI, Theorem 1.1). Thanks to the reductivity hypothesis, decompose the Lie algebra $\mathfrak{g}$ of G as

$$\mathfrak{g} = \mathfrak{g}^{ss} \oplus \mathfrak{a} = \mathfrak{p}^{ss} \oplus \mathfrak{k}^{ss} \oplus \mathfrak{a}^{\mathfrak{p}} \oplus \mathfrak{a}^{\mathfrak{k}},$$

where $\mathfrak{g}^{ss}$ is a semisimple ideal, $\mathfrak{a} = \mathfrak{z}(\mathfrak{g}) = \mathfrak{a}^{\mathfrak{k}} \oplus \mathfrak{a}^{\mathfrak{p}}$ is the center of $\mathfrak{g}$, $\mathfrak{k} = \mathfrak{k}^{ss} \oplus \mathfrak{a}^{\mathfrak{k}}$ is the Lie algebra of K, and $\mathfrak{p} = \mathfrak{p}^{ss} \oplus \mathfrak{a}^{\mathfrak{p}}$ is an $\mathrm{Ad}(K)$-invariant complement of $\mathfrak{k}$.

Definition 1.3.1. Let $\mathfrak{w}$ be an $\mathrm{Ad}(K)$-invariant vector subspace of $\mathfrak{g}$. We define $[\mathfrak{w}] \to N$ the fiber subbundle of the trivial bundle $N \times \mathfrak{g}$ whose fiber over $y \in N$ is

$$[\mathfrak{w}]_y = \mathrm{Ad}_y(\mathfrak{w}).$$

To be more precise, given $y \in N$, we should take $g \in G$ such that $y = gK$, and write $[\mathfrak{w}]_y = \mathrm{Ad}_g(\mathfrak{w})$. This is well-defined because of the invariance hypothesis.

Proposition 1.3.2. *The constant local system on N with fiber $\mathfrak{g}$ possesses a structure of even $\mathbb{R}$-phls $(\mathbb{V}_{\mathrm{ad}}, \sigma_{\mathrm{ad}}, Q_{\mathrm{ad}})$ given by*

$$(\mathcal{V}_{\mathrm{ad}}, D) = (N \times \mathfrak{g}, \mathrm{d}); \quad \sigma_{ad,y}(\xi) = \xi^{[\mathfrak{k}]} - \xi^{[\mathfrak{p}]};$$
$$Q_{\mathrm{ad}}(\xi, \xi) = -\mathrm{Kill}(\xi^{ss}, \xi^{ss}) + \|\xi^{\mathfrak{a}^{\mathfrak{k}}}\|^2 - \|\xi^{\mathfrak{a}^{\mathfrak{p}}}\|^2,$$

where $\xi^{[\mathfrak{k}]}$, $\xi^{[\mathfrak{p}]}$, ξ^{ss}, $\xi^{\mathfrak{a}^{\mathfrak{k}}}$, $\xi^{\mathfrak{a}^{\mathfrak{p}}}$ denote the projections of ξ on $[\mathfrak{k}]$, $[\mathfrak{p}]$, $\mathfrak{g}^{ss}$, $\mathfrak{a}^{\mathfrak{k}}$, $\mathfrak{a}^{\mathfrak{p}}$, respectively, Kill *is the Killing form of the real semisimple Lie algebra $\mathfrak{g}^{ss}$ and the squared norm is taken with respect to some fixed positive definite scalar product on $\mathfrak{a} \cong \mathbb{R}^n$.*

Proof. Notice that $\mathfrak{g}^{ss} = [\mathfrak{g}^{ss}] = [\mathfrak{k}] \oplus [\mathfrak{p}^{ss}]$ gives, at every point $y \in N$, a Cartan decomposition of the semisimple Lie algebra $\mathfrak{g}^{ss}$, which we define to be $\sigma_{ad,y}$. Hence

$$-\mathrm{Kill}\big(\xi, \sigma_{\mathrm{ad}}(\xi)\big) > 0 \quad \forall \xi \neq 0, \quad \xi \in \mathfrak{g}^{ss},$$

and the stated expression for S induces a symmetric, positive definite form on $\mathfrak{g}$.The only thing left to check is harmonicity; we claim that the map induced by the metric

$$f_{\mathrm{ad}} \colon N \to \mathrm{GL}(\mathfrak{g})/O(\mathfrak{g})$$

is even totally geodesic. That is because the map is induced by the inclusion of groups as inner automorphisms

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{GL}(\mathfrak{g}) \\ \big\downarrow & & \big\downarrow \\ N & \longrightarrow & \mathrm{GL}(\mathfrak{g})/O(\mathfrak{g}), \end{array}$$

which is well defined because $K < O(\mathfrak{g})$. To see that the map is as stated, fix a base point $y_0 = eK \in N$, so that we have a Cartan decomposition σ_0 and a reference metric $h_{y_0}(\xi, \xi) = -\mathrm{Kill}(\xi^{ss}, \sigma_0(\xi^{ss})) + \|\xi^{\mathfrak{a}}\|^2$. Then we have

$$h_y(\xi, \eta) = h_{y_0}\big(\mathrm{Ad}_y^{-1}(\xi), \mathrm{Ad}_y^{-1}(\eta)\big),$$

that by (1.4) means exactly that the harmonic map associates to $y \in N$ the adjoint automorphism Ad_y of $\mathfrak{g}$. ∎

Corollary 1.3.3. *Let M be a Riemannian manifold, $\rho\colon \Gamma = \pi_1(M) \to G$ a representation and $f\colon \tilde{M} \to N$ a ρ-equivariant and harmonic map. Then we can define on M a structure of even $\mathbb{R}$-phls by pulling back the adjoint structure on N.*

Proof. First start by constructing a $\mathbb{R}$-phls on $\tilde{M}$. All of the properties of a $\mathbb{R}$-phls pull back without problems, except possibly the harmonicity requirement. In this case, this follows from the proof of proposition 1.3.2, because f_{ad} is not only harmonic, but even totally geodesic, and composition of a harmonic map with a totally geodesic one is again harmonic.

To pass to M, notice that $f^*(\mathcal{V}_{\mathrm{ad}}, \mathrm{d}) = (\tilde{M} \times \mathfrak{g}, \mathrm{d})$, so we may define

$$\mathcal{V} = \tilde{M} \times_\Gamma \mathfrak{g},$$

where Γ acts on $\tilde{M}$ by deck transformations and on $\mathfrak{g}$ by the adjoint action Ad_ρ. This action commutes with d, so we have a flat bundle on M. The involution $f^*\sigma_{\mathrm{ad}}$ is defined by adjunction:

$$f^*\sigma_{\mathrm{ad}}(\xi)_{\tilde{x}} = \xi^{[\mathfrak{k}]_{f(\tilde{x})}} - \xi^{[\mathfrak{p}]_{f(\tilde{x})}},$$

so it passes to the quotient as well. The same is true for f^*S_{ad}, since it is induced by Kill, which is $\mathrm{Ad}(G)$-invariant. The metric on $\mathcal{V}$ is clearly given by $f \circ f_{\mathrm{ad}}$, as above, so it is harmonic. □

Fact 1.3.4. *Let $(\mathbb{V}, \sigma, Q)$ be a $\mathbb{R}$-phls on a Riemannian manifold M. Fix a point $\tilde{x} \in \tilde{M}$, and let $f\colon \tilde{M} \to \mathrm{GL}(n, \mathbb{R})/O(n)$ be the harmonic metric on $\mathcal{V}$. Let $\mathfrak{g}_0$ be the Lie algebra of G_0, defined as in notation 1.2.5. Then on $\tilde{M} \times_\Gamma \mathfrak{g}_0 \subseteq \tilde{M} \times_\Gamma \mathfrak{g}$, the two $\mathbb{R}$-phls structures induced by $f^*(\mathbb{V}_{\mathrm{ad}}, \sigma_{\mathrm{ad}}, Q_{\mathrm{ad}})$ and $\mathrm{End}(\mathbb{V}, \sigma, S)$ coincide.*

Analogously, if $(\mathbb{V}, \sigma, S)$ is a $\mathbb{C}$-phls and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$, and defining S_{ad} from Q_{ad} as in lemma 1.2.7, $f^*(\mathbb{V}_{\mathrm{ad}} \otimes \mathbb{C}, \sigma_{\mathrm{ad}} \otimes \mathbb{C}, S_{\mathrm{ad}})$ induces on $\tilde{M} \times_\Gamma \mathfrak{g}$ the same $\mathbb{C}$-phls structure as $\mathrm{End}(\mathbb{V}, \sigma, S)$.

Proof. We will first focus on the real, even case. Let $f_0\colon \tilde{M} \to N_0 = G_0/K_0 \subseteq \mathrm{GL}(n, \mathbb{C})/U(n)$ be a harmonic metric as in notation 1.2.5, so that $f = h \cdot f_0$ for some $h \in H$. Both the pull-back of the adjoint phls and $\mathrm{End}(\mathbb{V})$ live on isomorphic bundles:

$$\mathcal{V} \cong \tilde{M} \times_\Gamma \mathbb{R}^n \implies \mathrm{End}(\mathcal{V}) \cong \tilde{M} \times_\Gamma \mathfrak{gl}_n(\mathbb{R}).$$

The flat bundle $\tilde{M} \times_\Gamma \mathfrak{g}_0$ is a subbundle of this one. Note that f and f_0 induce the same metric on this bundle: Indeed, since $f = h \cdot f_0$ and h commutes with all of $\mathfrak{g}_0$, the metrics defined as in (1.4) coincide. By Corlette's theorem 1.1.1, the Lie algebra $\mathfrak{g}_0$ is reductive; write it as

$$\mathfrak{g}_0 = \bigoplus_i \mathfrak{g}_i \oplus \mathfrak{a},$$

where $\mathfrak{g}_i$ is simple and $\mathfrak{a}$ is abelian. We will prove that the two phls structures $f^*\mathbb{V}_{\text{ad}}$ and $\text{End}(\mathbb{V})$ induce the same positive definite metric h, and that the flat symmetric structures Q coincide on each factor $\mathfrak{g}_i$, up to possibly some constant factor λ_i. This will conclude the proof: First of all, the involution σ is uniquely determined on the abelian factor $\mathfrak{a}$, since it must be the identity on $\mathfrak{a}^{\mathfrak{k}} = \mathfrak{a} \cap \mathfrak{k}$. Secondly, let σ_1, σ_2 be the two involutions defined on a semisimple factor $\mathfrak{g}_i$ by the two different phls structures. Then, for every A, B in $\mathfrak{g}_i$, we obtain

$$h(A, B) = \lambda_i h\big(A, \sigma_1(\sigma_2(B))\big) \implies \sigma_1 \circ \sigma_2 = \lambda_i \cdot \text{Id} \implies \sigma_2 = \lambda_i \sigma_1.$$

This implies $\lambda_i = \pm 1$. But since at every point $\tilde{x} \in \tilde{M}$ both σ_1 and σ_2 have a Lie algebra as $+1$-eigenspace and something that is not closed under Lie bracket as -1-eigenspace, we must have $\lambda = 1$, hence all structures coincide.

The metric on $\text{End}(\mathcal{V})$ is induced by the composition

$$\tilde{M} \longrightarrow \text{GL}(n, \mathbb{R})/O(n) \times \text{GL}(n, \mathbb{R})/O(n) \xrightarrow{\otimes} \text{GL}(n^2, \mathbb{R})/O(n^2),$$

where the first map sends $\tilde{x}$ to $(A(\tilde{x}), A^{-1}(\tilde{x}))$, as in the proof of lemma 1.2.8. On the other hand, the metric on $f^*\mathcal{V}_{\text{ad}}$ is induced by the adjoint action:

$$\tilde{M} \xrightarrow{f} \text{GL}(n, \mathbb{R})/O(n) \to \text{GL}(\mathfrak{gl}_n(\mathbb{R}))/U(\mathfrak{gl}_n(\mathbb{R})).$$

These two maps amount to the same thing: Denoting by $e_i \otimes e_j$ the matrix with one 1 at the place (i, j) only, we have, for the first definition:

$$(e_i \otimes e_j)(A \otimes A^{-1})(e_k \otimes e_h) = A_{ik} \cdot A^{jh},$$

and for the second one

$$\big\langle (e_i \otimes e_j), A \cdot (e_k \otimes e_h) \cdot A^{-1} \big\rangle = A_{ik} \cdot A^{jh}.$$

The equality of the S's on each $\mathfrak{g}_i$, up to a constant factor, is easy, using the definition with f_0 instead of f: Both S_{End} and $f_0^* S_{\text{ad}}$ give, on each point $\tilde{x} \in \tilde{M}$, a symmetric and, by the real version of corollary 1.2.4, $\mathfrak{g}_0$-invariant form on $\mathfrak{g}_0$, so it is the Killing form on each factor, up to some multiple. These multiples must be constant in $\tilde{x}$ because, by hypothesis, both forms are flat.

The complex case (hence, the odd real one) is treated analogously, with only slightly more attention needed. Indeed, since $\mathfrak{g}_0 \subset \mathfrak{gl}_n(\mathbb{C})$, we may suppose $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{C})$, so the equality of the positive definite metrics works just as well, substituting everywhere $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{R})$ and $U(n)$ to $O(n)$ above. For the flat quadratic forms, in this case we are given two hermitian forms on $\mathfrak{g}$, S_1 and S_2, which are only $\mathfrak{g}_0$-invariant. However, the same method works: Since $\mathfrak{g}_0$ is reductive, $\mathfrak{g}$ is, hence we reduce as above to the case where $\mathfrak{g}$ is simple. Then, the two hermitian matrices representing S_1 and S_2 can be simultaneously diagonalized, hence there is a λ such that $\mathrm{Ker}(S_1 - \lambda S_2) \neq 0$. The application $S_1 - \lambda S_2$ is only $\mathfrak{g}_0$-invariant, but clearly this implies that its kernel is $\mathfrak{g}_0 \otimes \mathbb{C}$-invariant, hence a non-empty ideal of $\mathfrak{g}$, which must then coincide with $\mathfrak{g}$. In the real, odd case, the equality of S_1 and S_2 implies that between Q_1 and Q_2. □

1.4 The tangent bundle of a symmetric space

Recall that there is a canonical surjective map of vector bundles:

$$\begin{aligned} \vartheta_{TN} \colon N \times \mathfrak{g} &\to TN \\ (x, \xi) &\mapsto \Big(x, \frac{\partial}{\partial t}(\exp(t\xi) \cdot x)\big|_{t=0}\Big). \end{aligned} \tag{1.5}$$

It is easy to see that the kernel of ϑ_{TN} is $[\mathfrak{k}]$. Thus the restriction to the complement

$$\vartheta_{TN}\big|_{[\mathfrak{p}]} \colon [\mathfrak{p}] \to TN$$

is an isomorphism of vector bundles over N. Henceforth, the inverse of this isomorphism will be denoted by β_N, which is thus a 1-form $\beta_N \in \mathcal{A}^1_N(\mathfrak{g})$ inducing on every stalk the isomorphism of vector spaces $T_yN \cong [\mathfrak{p}]_y$.

Definition 1.4.1. The 1-form β_N defined above is called the Maurer-Cartan form of the symmetric space N.

The name comes from the more general construction which holds on any reductive homogeneous space $N' = G/H$: Taking $N' = G$, then $\beta_{N'}$ coincides with the usual right Maurer-Cartan form (cfr. [BR90], chapter 1).

One can easily check equivariance for the 1-form β_N:

$$g^*\beta_N = \mathrm{Ad}_g \circ \beta_N. \tag{1.6}$$

In 1.3 we defined a phls structure on $N \times \mathfrak{g}$. We may restrict this to the subbundle $[\mathfrak{p}] \to N$, which is thus endowed with a metric. Thanks to ϑ_{TN} and β_N, this gives TN a G-invariant metric.

The total space of the tangent bundle TN of a Riemannian symmetric space N is not a Riemannian symmetric space itself when N is not flat, see [Kow71]. On the other hand, it is naturally a homogeneous space, as proved in [BS72]. We resume this result in the following, adapting notations to our setting:

Proposition 1.4.2. *The total space of the tangent bundle TN is a homogeneous space with transformation group TG. Explicitly, if we define the diffeomorphism:*

$$\begin{aligned} r\colon TG &\to G \ltimes \mathfrak{g} \\ (g, v) &\mapsto \big(g, R_{g^{-1}*}v\big), \end{aligned}$$

we obtain a commutative diagram

$$\begin{array}{ccc} TG & \xrightarrow{\;r\;} & G \ltimes \mathfrak{g} \\ \Big\downarrow & & \Big\downarrow \\ {\scriptstyle \mathrm{d}\pi_N} & & N \times \mathfrak{g} \\ \Big\downarrow & \swarrow{\scriptstyle \vartheta_{TN}} & \\ TN & & \end{array} \tag{1.7}$$

where the vertical arrow on the right is quotient by K either seen as a subgroup of TG or equivalently by acting by right multiplication on G and adjunction on $\mathfrak{g}$.

Remark 1.4.3. We adopted the semidirect notation $G \ltimes \mathfrak{g}$ because the multiplication induced by r on $G \ltimes \mathfrak{g}$ is exactly

$$(g, \xi) \cdot (h, \eta) = \big(gh, \xi + \mathrm{Ad}_g(\eta)\big).$$

Remark 1.4.4. Since we are assuming that G is the group of real points of a connected reductive algebraic group $\mathbb{G}$, in the analysis of the tangent space we can take the algebraic perspective as well. First of all, since K is maximal compact, it intersects every connected component of G, so that G/K is connected. Secondly, by the Weyl unitarian trick, K consists of the elements of G which are orthogonal for some positive definite symmetric form q. Thus, letting $\mathbb{K} = \mathbb{G} \cap O(q)$, we have $K = \mathbb{K}(\mathbb{R})$. Denote by $\mathbf{N} = \mathbb{G}/\mathbb{K}$ the scheme-theoretic quotient. We have an equality $\mathbf{N}(\mathbb{R})^\circ = G/K = N$, where $\mathbf{N}(\mathbb{R})^\circ$ is the neutral connected component of $\mathbf{N}(\mathbb{R})$. Then the connected component of $\mathbb{R}[\varepsilon]/(\varepsilon^2)$-points of $\mathbf{N}$ coincides with the tangent space:

$$TN = \mathbf{N}\big(\mathbb{R}[\varepsilon]/(\varepsilon^2)\big)^\circ = TG/TK = \mathbb{G}\big(\mathbb{R}[\varepsilon]/(\varepsilon^2)\big)\Big/\mathbb{K}\big(\mathbb{R}[\varepsilon]/(\varepsilon^2)\big).$$

1.5 The canonical connection

We are going to introduce a connection on any complex phls (but everything works equally well for real ones), the canonical connection, which is metric with respect to h and depends on f only. For a start, we define a canonical 1-form $\beta \in \mathcal{A}^1_M(\mathcal{V})$, which is essentially the differential of f.

Definition 1.5.1. Let $(\mathcal{V}, D, h)$ be a complex harmonic bundle on M, and $f\colon \tilde{M} \to P = \mathrm{GL}(n,\mathbb{C})/U(n)$ as in (1.4). We define the 1-form $\tilde{\beta} \in \mathcal{A}^1_{\tilde{M}}(\mathfrak{gl}_n(\mathbb{C}))$ as

$$\tilde{\beta} = f^* \beta_P,$$

and $\beta \in \mathcal{A}^1_M(\mathrm{End}(\mathcal{V}))$ the 1-form such that $\tilde{\beta} = \tilde{\pi}^* \beta$.

Remark 1.5.2. The 1-form $\tilde{\beta}$ is the pull-back of an $\mathrm{End}(\mathcal{V})$-valued 1-form β on M, because, thanks to (1.6), $\tilde{\beta}$ is equivariant:

$$\tilde{\beta}_{\gamma\tilde{x}} = \big(\rho(\gamma)\cdot f(\tilde{x})\big)^* \beta_P = f^*\big(\rho(\gamma)^*\beta_P\big) = \mathrm{Ad}_{\rho(\gamma)}\tilde{\beta}_{\tilde{x}}.$$

First of all, remark that if we have a reductive subgroup $G \subseteq \mathrm{GL}(n,\mathbb{C})$ such that $f\colon \tilde{M} \to N = G/K \subseteq P = \mathrm{GL}(n,\mathbb{C})/U(n)$, then we can equivalently define $\tilde{\beta}$ as the pullback of β_N:

Lemma 1.5.3. *Let $G < G'$ be an inclusion of reductive algebraic Lie groups, K a maximal compact subgroup of G, and extend it to a maximal compact subgroup K' of G'. Then, denoting by $i\colon N = G/K \to N' = G'/K'$ the totally geodesic embedding, $i^*\beta_{N'} = \beta_N$.*

Proof. This follows easily from the compatibility of the exponential mappings. □

For the special case of the adjoint phls, then, since the harmonic map is given by adjunction, the canonical 1-form $\beta \in \mathcal{A}^1_N(\mathrm{End}(\mathfrak{g}))$ is just β_N under the adjoint representation, that is, for every tangent vector field Y on N and $\xi \in \mathfrak{g}$,

$$\beta(Y)(\xi) = [\beta_N(Y), \xi]. \tag{1.8}$$

Corollary 1.5.4. *Let $(\mathbb{V}, \sigma, S)$ be a phls on M, f a harmonic metric, and β the corresponding 1-form. Then the 1-form $\beta^{\mathrm{End}} \in \mathcal{A}^1\big(\mathrm{End}(\mathrm{End}(\mathcal{V}))\big)$ corresponding to $\mathrm{End}(\mathbb{V}, \sigma, S)$ is $\mathrm{ad}(\beta)$, that is, for every $A \in \mathrm{End}(\mathcal{V})$, we have*

$$\beta^{\mathrm{End}}(A) = \beta \circ A - A \circ \beta.$$

Proof. It is enough to prove this when $f = f_0$, as in notation 1.2.5, so that β takes values in $\mathfrak{g}_0$, since if $f = hf_0$, then β changes only by adjunction by h. By fact 1.3.4, the pullbacks $\tilde{\pi}^*\mathrm{End}(\mathbb{V}, \sigma, S)$ and $f^*(\mathbb{V}_{\mathrm{ad}}, \sigma_{\mathrm{ad}}, S_{\mathrm{ad}})$ coincide on $\tilde{M} \times_\Gamma \mathfrak{g}_0$ (or its complexification, in the $\mathbb{C}$-phls case). By (1.8), the 1-form associated to the adjoint phls on $N = \mathrm{GL}(n, \mathbb{C})/U(n)$ is $\mathrm{ad}(\beta_N)$, and the pull-back of β_N to $\tilde{M}$ is $\tilde{\beta}$, by definition. □

Lemma 1.5.5. *The 1-form β takes values in $\mathrm{End}(\mathcal{V})^-$, the (-1)-eigenspace of $\sigma_{\mathrm{End}(\mathcal{V})}$ (which is the subspace exchanging $\mathcal{V}^+$ and $\mathcal{V}^-$).*

Proof. This follows from fact 1.3.4. Indeed,

$$\beta_N \colon T\big(\mathrm{GL}(n, \mathbb{C})/U(n)\big) \xrightarrow{\sim} [\mathfrak{p}] = \mathcal{V}^-_{\mathrm{ad}},$$

where $\mathfrak{p} \subset \mathfrak{gl}_n$ is the subspace of symmetric or Hermitian matrices, hence $\sigma_{\mathrm{ad}}(\beta_N(Y)) = -\beta_N(Y)$ for every $Y \in T(\mathrm{GL}(n, \mathbb{C})/U(n))$. Composing with $\mathrm{d}f$ gives the result. □

Definition 1.5.6. Let $(\mathbb{V}, \sigma, S)$ be a phls on M, $(\mathcal{V}, D)$ the associated flat bundle and f a harmonic metric. Let v be a section of $\mathcal{V}$. We define the *canonical connection* on $(\mathbb{V}, \sigma, S)$ by any of the following two expressions:

1. $\mathrm{d}^{\mathrm{can}} v = \big(D(v^+)\big)^+ + \big(D(v^-)\big)^-$;
2. $\mathrm{d}^{\mathrm{can}} v = Dv - \beta \cdot v$.

Here, v^+ and v^- are the projections of v on $\mathcal{V}^+$ and $\mathcal{V}^-$, respectively, and $\beta \cdot v \in \mathcal{A}^1(\mathcal{V})$ is defined by applying the $\mathrm{End}(\mathcal{V})$-part of β to v. We will denote by $\overset{\mathrm{can}}{\nabla} \colon \mathcal{A}^p(\mathcal{V}) \to \mathcal{A}^{p+1}(\mathcal{V})$ the covariant derivative of tensors, and by $\mathrm{d}^{\mathrm{can}} \colon \mathcal{A}^p(\mathcal{V}) \to \mathcal{A}^{p+1}(\mathcal{V})$ the exterior covariant derivative, that is, the anti-symmetrization of $\overset{\mathrm{can}}{\nabla}$ (they coincide for $p = 0$).

Proposition 1.5.7. *The two definitions in 1.5.6 coincide, and define a metric connection for h.*

Lemma 1.5.8. *Let $(\mathbb{V}, \sigma, S)$ be a phls on M, $x \in M$ a point, $\rho \colon \pi_1(M) \to G_0 \subseteq \mathrm{GL}(n, \mathbb{C})$ the monodromy representation. Let $f \colon \tilde{M} \to \mathrm{GL}(n, \mathbb{C})/U(n)$ be a harmonic metric, and $s \colon \tilde{M} \to \mathrm{GL}(n, \mathbb{C})$ a lift of f. Then*

$$\sigma_0 \overset{\mathrm{def}}{=} s^{-1} \circ \sigma \circ s$$

is flat, i.e. $D(\sigma_0) = 0$.

Proof. This follows easily from the existence of an $f_0 \colon \tilde{M} \to N_0$ such that $f = h \cdot f_0$, as in notation 1.2.5. Work with pull-backs to $\tilde{M}$ for ease, and let $s_0 \colon \tilde{M} \to G_0$ be a section of f_0. In particular, $s(\tilde{x}) = h \cdot s_0(\tilde{x}) = s_0(\tilde{x}) \cdot h$. If we prove that $s_0^{-1} \circ \sigma \circ s_0$ is flat, then the same is true for $\sigma_0 = h^{-1} \circ s_0^{-1} \circ \sigma \circ s_0 \circ h$. Hence, from now on, we suppose that $f = f_0$ so that s takes values in G_0.

Let v, w be any two sections of $\tilde{\mathcal{V}} = \tilde{M} \times \mathbb{C}^n$. Then, by the definition of f in (1.4),

$$h(v, w) = S\big(v, \sigma(w)\big) = \big\langle s^{-1}v, s^{-1}w \big\rangle_{\mathbb{C}^n}.$$

In terms of $v' = s^{-1}v$ and $w' = s^{-1}w$, we obtain, thanks to corollary 1.2.4,

$$S\big(v', \sigma_0(w')\big) = S\Big(v', s^{-1} \cdot \big(\sigma(sw')\big)\Big) = S\Big(sv', \sigma(sw')\Big) = \big\langle v', w' \big\rangle_{\mathbb{C}^n}.$$

Now, by hypothesis S is flat, and the constant scalar product on $\mathbb{C}^n$ is clearly flat as well. Since v' and w' are arbitrary functions $\tilde{M} \to \mathbb{C}^n$, the involution σ_0 must be flat, too. □

Lemma 1.5.9. *With the same notations of the preceding lemma, let $N = G/K$ be any totally geodesic subspace of $\mathrm{GL}(n, \mathbb{C})/U(n)$ such that f takes values in N, so that $s \colon \tilde{M} \to G \subseteq \mathrm{GL}(n, \mathbb{C})$. Let $\alpha = s^*\theta_r = \mathrm{d}s \cdot s^{-1}$ be the pull-back of the Maurer-Cartan form θ_r of G, which is thus a $\mathfrak{g} = \mathrm{Lie}(G)$-valued 1-form on $\tilde{M}$. Then:*

1. *$\tilde{\beta} = \alpha^{[\mathfrak{p}]}$, where $\mathfrak{p}$ is the subspace of $\mathfrak{g}$ on which σ_{ad} is negative definite and $[\mathfrak{p}]_{\tilde{x}} = \mathrm{Ad}_{s(\tilde{x})}(\mathfrak{p})$.*

2. *Let v be a section of $\tilde{\mathcal{V}}$, the pull-back of $\mathcal{V}$ to $\tilde{M}$. Then*

$$\nabla^\alpha v \overset{\mathrm{def}}{=} Dv - \alpha \cdot v = s \cdot \big(D(s^{-1}v)\big).$$

3. *The connection ∇^α is metric for π^*h and sends sections of $\tilde{\mathcal{V}}^+$ to $\tilde{\mathcal{V}}^+$-valued 1-forms, and similarly for $\tilde{\mathcal{V}}^-$.*

Proof. 1. We will just prove this when $G = \mathrm{GL}(n, \mathbb{C})$ but the proof is exactly the same in the general case (actually, lemma 1.5.3 tells us that proving this case is enough). We know that $\tilde{\beta}$ takes values in the subbundle $[\mathfrak{gl}_n(\mathbb{C})^-]$ of $\tilde{M} \times \mathfrak{gl}_n(\mathbb{C})$, where $f^*\sigma_{\mathrm{ad}}$ is negative definite; to prove that

$$\tilde{\beta} = f^*\beta_N = s^*p^*(\beta_N) = \alpha^- = s^*(\theta_r^-),$$

where θ_r^- is the projection on $[\mathfrak{gl}_n(\mathbb{C})^-]$ of θ_r, we prove directly the equality $p^*\beta_{N_0} = \theta_r^-$. Equivalently, taking the composition with ϑ_{TN}, we shall prove that

$$p_*(X) = \vartheta_{TN} \circ \theta_r^-(X), \quad \forall X \in TG.$$

Both β_N and θ_r are $\mathrm{Ad}(G)$-equivariant, hence it suffices to prove this in the point $e \in G$. But there ϑ_{TN} is p_* by definition, and θ_r is the identity.

2. This is a simple computation:

$$s \cdot D(s^{-1} \cdot v) = Dv - s \cdot s^{-1} \cdot \mathrm{d}(s) \cdot s^{-1} \cdot v = Dv - \alpha \cdot v.$$

3. This follows from lemma 1.5.8. Since σ_0 is D-flat, we have also that σ is $s \circ D \circ s^{-1}$-flat, that is, ∇_α commutes with σ. In particular, it respects the decomposition $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}^+ \oplus \tilde{\mathcal{V}}^-$. It also respects the metric:

$$\begin{aligned} \mathrm{d}h(v, w) &= \mathrm{d}\langle s^{-1}v, s^{-1}w\rangle_{\mathbb{C}^n} = \big\langle \mathrm{d}(s^{-1}v), s^{-1}w\big\rangle_{\mathbb{C}^n} + \big\langle s^{-1}v, \mathrm{d}(s^{-1}w)\big\rangle_{\mathbb{C}^n} \\ &= h\big((s \circ D \circ s^{-1}(v), w\big) + h\big(v, s \circ D \circ s^{-1}(w)\big). \end{aligned}$$

□

Lemma 1.5.10. *Let* $(\mathbb{V}, \sigma, S)$ *be a phls on* M*. Then, with respect to the metric structure* $h = S(\cdot, \sigma(\cdot))$*, the decomposition*

$$\mathrm{End}(\mathcal{V}) = \mathrm{End}(\mathcal{V})^+ \oplus \mathrm{End}(\mathcal{V})^-$$

coincides with the decomposition into anti self-adjoint and self-adjoint endomorphisms, respectively.

Proof. We prove that an element $A \in \mathrm{End}(\mathcal{V})^+$ is anti self-adjoint, the other part being identical. Write $f = h \cdot f_0$, with $f_0 \colon \tilde{M} \to G_0/K_0$. Then the structure given by f_0 defines another decomposition $\mathcal{V} = \mathcal{V}^{+_0} \oplus \mathcal{V}^{-_0}$, hence a decomposition of $\mathrm{End}(\mathcal{V})$, such that

$$\mathrm{End}(\mathcal{V})^+ = \mathrm{Ad}_h \mathrm{End}(\mathcal{V})^{+_0}.$$

Thanks to fact 1.3.4, we know that $\mathrm{End}(\mathcal{V})^{+_0}$ is the pull-back under f_0 of $[\mathfrak{gl}_n(\mathbb{C})^+]$, that is, for every A_0 therein, $s_0^{-1} \cdot A_0 \cdot s_0 \in \mathfrak{gl}_n(\mathbb{C})^+$ is an anti-hermitian matrix. It follows that for every A in $\mathrm{End}(\mathcal{V})^+$, also $s^{-1} \cdot A \cdot s$ is anti-hermitian. The result follows by the definition of h with respect to s:

$$h(Av, w) = \big\langle s^{-1}Av, s^{-1}w\big\rangle_{\mathbb{C}^n} = \big\langle s^{-1}As \cdot s^{-1}v, s^{-1}w\big\rangle_{\mathbb{C}^n} = -h(v, Aw).$$

□

Proof of proposition 1.5.7. First we prove that the second definition of $\mathrm{d}^{\mathrm{can}}$ respects the $\mathcal{V}^+ \oplus \mathcal{V}^-$ decomposition: Indeed, thanks to point 1. of lemma 1.5.9,

$$\mathrm{d}^{\mathrm{can}}v = Dv - \beta \cdot v = Dv - \alpha \cdot v + \alpha^+ \cdot v = \nabla_\alpha + \alpha^+ \cdot v;$$

now the first summand respects the decomposition thanks to point 3. of lemma 1.5.9, and the second one does too, because by definition α^+ takes value in $\mathrm{End}(\mathcal{V})^+$.

Now take, for example, a section v of $\mathcal{V}^+$; then, the two definitions yield

$$\begin{aligned} \mathrm{d}^{\mathrm{can}} v &= (\mathrm{d}v)^+; \\ \mathrm{d}^{\mathrm{can}} v &= \mathrm{d}v - \beta \cdot v. \end{aligned} \tag{1.9}$$

To prove that they are equal, simply observe that, thanks to lemma 1.5.5, $\beta \cdot v$ is a $\mathcal{V}^-$-valued 1-form. Hence, (1.9) being a $\mathcal{V}^+$-valued 1-form thanks to the above discussion, it must coincide with the projection on $\mathcal{V}^+$ of $\mathrm{d}v$, that is, the first definition.

Finally, $\mathrm{d}^{\mathrm{can}}$ is a metric connection, because point 1. of lemma 1.5.9 implies that it differs from the metric connection ∇_α only by an $\mathrm{End}(\mathcal{V})^+$-valued form, which is anti self-adjoint by lemma 1.5.10. □

Remark 1.5.11. The origin of the name "canonical connection" is that, in the case when $\tilde{M} = N$ is a symmetric space equipped with the adjoint phls, the connection thus defined restricts, on the subbundle $[\mathfrak{p}] \subseteq N \times \mathfrak{g}$, to the canonical connection of N, which can be defined in several ways, for example as the Levi-Civita connection associated to any invariant metric on N, or as the principal connection on the K-bundle $G \to N$ determined by $\mathfrak{p}$ (cfr. [KN63], pag. 302, [Hel78], pag 217 and [BR90], Chapter 1).

Lemma 1.5.12. *Let $(\mathcal{V}, D)$ be a flat bundle with a phls structure. Let ϕ be a $\mathcal{V}$-valued p-form. Then, if we write* d *for the flat exterior differential associated to D, the exterior differential* $\mathrm{d}^{\mathrm{can}}\phi$ *can be written in either of the following ways:*

$$\begin{aligned} \mathrm{d}^{\mathrm{can}}\phi &= \big(\mathrm{d}(\phi^+)\big)^+ + \big(\mathrm{d}(\phi^-)\big)^-; \\ \mathrm{d}^{\mathrm{can}}\phi &= \mathrm{d}\phi - \beta \wedge \phi. \end{aligned}$$

Proof. This is just a direct computation with the definition of exterior differential. We only notice that the definition of $\beta \wedge \phi$ can be taken to be

$$\beta \wedge \phi(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \beta(X_i) \cdot \phi(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}).$$

Here $\cdot$ denotes the action of $\mathrm{End}(\mathcal{V})$ on $\mathcal{V}$. □

Corollary 1.5.13. *Let $(\mathbb{V}, \sigma, S)$ be a phls with a harmonic metric f, and β the associated 1-form. Then β is parallel with respect to the canonical connection on* $\mathrm{End}(\mathbb{V}, \sigma, S)$. *Explicitly, we have the Maurer-Cartan equation:*

$$\mathrm{d}\beta = [\beta, \beta] \tag{1.10}$$

(notice the absence of a factor $\frac{1}{2}$ with respect to the usual Maurer-Cartan equation on a Lie group).

Proof. By the lemma above plus corollary 1.5.4, equation (1.10) is exactly $\mathrm{d}^{\mathrm{can}}\beta = 0$. To prove this parallelness, work on the universal cover $\tilde{M}$ and compute $\mathrm{d}^{\mathrm{can}}\alpha$, where α is the pull-back of the right Maurer-Cartan form, as in lemma 1.5.9:

$$\mathrm{d}^{\mathrm{can}}\alpha = \mathrm{d}\alpha - [\alpha^{[\mathfrak{p}]}, \alpha] = \frac{1}{2}[\alpha, \alpha] - [\alpha^{[\mathfrak{p}]}, \alpha] = \frac{1}{2}[\alpha^{[\mathfrak{k}]}, \alpha^{[\mathfrak{k}]}] - \frac{1}{2}[\alpha^{[\mathfrak{p}]}, \alpha^{[\mathfrak{p}]}].$$

This proves that $\mathrm{d}^{\mathrm{can}}\alpha$ is a $[\mathfrak{k}]$-valued 2-form, and since $\mathrm{d}^{\mathrm{can}}\beta = (\mathrm{d}^{\mathrm{can}}\alpha)^{[\mathfrak{p}]}$, it follows that $\mathrm{d}^{\mathrm{can}}\beta = 0$. □

Thanks to this expression, one can easily compute the curvature of the canonical connection:

$$R^{can}(\cdot, \cdot)v = -[\beta, \beta] \cdot v + \beta \cdot (\beta \cdot v). \tag{1.11}$$

In the special case of the adjoint phls on a symmetric space, where $\beta \cdot v = [\beta, v]$ etc., this clearly reduces to the usual curvature formula for a symmetric space via the Jacobi identity:

$$R^{can}(\cdot, \cdot)v = \frac{1}{2}\big[[\beta, \beta], v\big]. \tag{1.12}$$

1.6 The codifferential and the Laplacian

Let $(\mathbb{V}, \sigma, S)$ be a (complex or real) phls structure on M, which we again suppose compact. The purpose of this section is to compute the codifferential d^* of the connection D with respect to the metric h on 1-forms, and deduce a formula for the Laplacian $\Delta = \mathrm{d}^*\mathrm{d}$ on sections. We will always work with a complex phls, but everything translates without problems in the real setting.

Definition 1.6.1. Define the connection $\tilde{\nabla}$ on $\mathcal{V}$ by:

$$\tilde{\nabla} v = \overset{\mathrm{can}}{\nabla} v - \beta \cdot v = Dv - 2\beta \cdot v, \quad \forall v \in \mathcal{C}^\infty(\mathcal{V}).$$

Again, denote by $\tilde{\mathrm{d}}$ the associated exterior differential.

Remark 1.6.2. When $\mathbb{V} = \mathbb{V}_{\mathrm{ad}}$ is the adjoint phls structure on a symmetric space N, then $\tilde{\nabla}$ is flat. To see this, first apply the definition of $\tilde{\mathrm{d}}$ to get

$$\begin{aligned}\tilde{\mathrm{d}}^2 v &= \tilde{\mathrm{d}}\Big(\mathrm{d}v - 2[\beta, v]\Big)\\ &= -2[\mathrm{d}\beta, v] + 2[\beta, \mathrm{d}v] - 2[\beta, \mathrm{d}v] + 4\big[\beta, [\beta, v]\big].\end{aligned}$$

Then, using (1.10) and the graded Jacobi identity:

$$\begin{aligned}\tilde{\mathrm{d}}^2 v &= -2\big[[\beta,\beta],v\big] + 4\big[\beta,[\beta,v]\big] \\ &= 2\big[v,[\beta,\beta]\big] + 2\big[\beta,[\beta,v]\big] - 2\big[\beta,[v,\beta]\big] = 0.\end{aligned}$$

Lemma 1.6.3. *Let α be a $\mathcal{V}$-valued 1-form. Let $x \in M$ be a point, and denote by $\{E_s\}$ a local frame, and g^{st} the inverse matrix of the metric. Then:*

$$\mathrm{d}^*\alpha = \mathrm{d}^{\mathrm{can}*}\alpha + \sum_{s,t} g^{st}\beta(E_s)\cdot\alpha(E_t) = -\mathrm{trace}\big(\tilde{\nabla}\alpha\big).$$

Proof. By definition of the codifferential and of $\mathrm{d}^{\mathrm{can}}$, for every section v of $\mathcal{V}$,

$$\begin{aligned}\int_M \langle \mathrm{d}^*\alpha, v\rangle \mathrm{dVol}_g &= \int_M \langle \alpha, \mathrm{d}v\rangle \mathrm{dVol}_g \\ &= \int_M \langle \alpha, \mathrm{d}^{\mathrm{can}} v\rangle \mathrm{dVol}_g + \int_M \langle \alpha, \beta\cdot v\rangle \mathrm{dVol}_g \\ &= \int_M \langle \mathrm{d}^{\mathrm{can}*}\alpha, v\rangle \mathrm{dVol}_g + \int_M \langle \alpha, \beta\cdot v\rangle \mathrm{dVol}_g\end{aligned}$$

Now, locally $\mathrm{d}^{\mathrm{can}*}\alpha = -\sum_{s,t} g^{st} \overset{\mathrm{can}}{\nabla}_{E_t}\alpha(E_s)$, by (1.1), and

$$\langle \alpha, \beta\cdot v\rangle = \sum_{s,t} g^{st}\langle \alpha(E_s), \beta(E_t)\cdot v\rangle = \sum_{s,t} g^{st}\langle \beta(E_s)\cdot\alpha(E_t), v\rangle,$$

where in the last equality we have used that $\beta(E_s)$ is self-adjoint, thanks to lemmas 1.5.5 and 1.5.10. □

This gives a formula for the codifferential of $\mathcal{V}$-valued 1-forms. Pulling back to $\tilde{M}$, we obtain a formula for the codifferential of ρ-equivariant $\tilde{\mathcal{V}}$-valued forms $\tilde{\alpha} = \tilde{\pi}^*\alpha$. We extend these definitions to every $\tilde{\mathcal{V}} = \tilde{M}\times\mathbb{C}^n$-valued 1-form:

Definition 1.6.4. Let $\tilde{\alpha} \in \mathcal{A}^1(\tilde{M}\times\mathbb{C}^n)$ be a 1-form and $v\colon \tilde{M}\to\mathbb{C}^n$ a smooth function (that is, a section of $\tilde{\mathcal{V}}$). We define, in terms of a local frame $\{E_s\}$,

$$\mathrm{d}^{\mathrm{can}*}\alpha = -\mathrm{trace}\overset{\mathrm{can}}{\nabla}\alpha, \qquad \mathrm{d}^*\alpha = \mathrm{d}^{\mathrm{can}*}\alpha + \sum_{s,t} g^{st}\big([\tilde{\beta}(E_s), \tilde{\alpha}(E_t)]\big),$$

$$\Delta^{\mathrm{can}}(v) = \mathrm{d}^{\mathrm{can}*}\mathrm{d}^{\mathrm{can}} v, \qquad \Delta(v) = \mathrm{d}^*\mathrm{d}v.$$

Definition 1.6.5. Let $v\colon \tilde{M} \to \mathbb{C}^n$ be a section of $\tilde{\mathcal{V}} = \tilde{M} \times \mathbb{C}^n$. We define the *Jacobi operator* as

$$J(v) = \Delta^{\mathrm{can}}(v) + \sum_{s,t} g^{st} \tilde{\beta}(E_s) \cdot (\tilde{\beta}(E_t) \cdot v).$$

This object has been introduced in a different context for the study of variations of non-twisted harmonic maps, cfr. [EL83, Maz73], and will be central in the next chapters. The following useful statement, and its corollaries, is the reason we introduced phls.

Lemma 1.6.6. *Let $v\colon \tilde{M} \to \mathbb{C}^n$ be a section of $\tilde{\mathcal{V}} = \tilde{M} \times \mathbb{C}^n$. Then we have the following Weitzenböck formula:*

$$\mathrm{d}^* \mathrm{d} v = J(v).$$

Proof. Denote by E_s a local orthonormal system near a point $\tilde{x} \in \tilde{M}$. Then

$$\begin{aligned}
\mathrm{d}^*\mathrm{d}v &= -\sum_s \Big(\overset{\mathrm{can}}{\nabla}_{E_s} D_{E_s} v - \tilde{\beta}(E_s) \cdot D_{E_s} v \Big) \\
&= \sum_s - \overset{\mathrm{can}}{\nabla}_{E_s} \overset{\mathrm{can}}{\nabla}_{E_s} v - \overset{\mathrm{can}}{\nabla}_{E_s} \big(\tilde{\beta}(E_s) \cdot v \big) + \tilde{\beta}(E_s) \cdot \overset{\mathrm{can}}{\nabla}_{E_s} v + \tilde{\beta}(E_s) \cdot \big(\tilde{\beta}(E_s) \cdot v \big) \\
&= \Delta^{\mathrm{can}} v + \sum_s \tilde{\beta}(E_s) \cdot \big(\tilde{\beta}(E_s) \cdot v \big) - \overset{\mathrm{can}}{\nabla}_{E_s} (\tilde{\beta}(E_s)) \cdot v,
\end{aligned}$$

where in the last equality we have used lemma 1.6.7 below. Finally, we claim that

$$\sum_s \overset{\mathrm{can}}{\nabla}_{E_s} \tilde{\beta}(E_s) = 0$$

is in fact equivalent to the metric f being harmonic. First recall from (1.2) that f is harmonic if and only if its tension field $\tau(f) = \sum_s \overset{N}{\nabla}_{E_s} \mathrm{d}f(E_s)$ vanishes, where $\overset{N}{\nabla}$ is the Levi-Civita connection of N. Now, denoting by $P = \mathrm{GL}(n, \mathbb{C})/U(n)$ and by β_P its Maurer-Cartan form, we compute:

$$\beta_P(\tau(f)) = \beta_P \Big(\sum_s \overset{N}{\nabla}_{E_s} (\mathrm{d}f(E_s)) \Big) = \sum_s \overset{\mathrm{can}}{\nabla}_{E_s} \tilde{\beta}(E_s),$$

where we have used fact 1.3.4, the fact that for the adjoint phls the canonical connection corresponds to the Levi-Civita connection, and the definition of $\tilde{\beta}(E_s) = f^* \beta_P(E_s)$. Since β_P is injective, this concludes the proof. □

Lemma 1.6.7. *The canonical connection is compatible with the product, that is, if v is a section of $\mathcal{V}$ and A is a section of* $\mathrm{End}(\mathcal{V})$*, we have*

$$\mathrm{d}^{\mathrm{can}}(A \cdot v) = \mathrm{d}^{\mathrm{can}}(A) \cdot v + A \cdot \mathrm{d}^{\mathrm{can}}(v).$$

Proof. We prove this after pull-back to $\tilde{M}$. Using $\mathrm{d}^{\mathrm{can}} v = \mathrm{d}v - \tilde{\beta} \cdot v$, and $\mathrm{d}^{\mathrm{can}} A = \mathrm{d}A - [\tilde{\beta}, A]$, plus the trivial relation $\mathrm{d}(A \cdot v) = \mathrm{d}A \cdot v + A \cdot \mathrm{d}v$, we obtain the result. □

Corollary 1.6.8. *The Laplacian $\Delta = \mathrm{d}^*\mathrm{d}$ on a phls respects the decomposition*

$$\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-.$$

Proof. Indeed, in the definition of J, which we have just proved to coincide with Δ, the Laplacian Δ^{can} respects the decomposition, since $\mathrm{d}^{\mathrm{can}}$ does, and $\tilde{\beta}$, which exchanges the factors of the decomposition, is applied twice. □

Corollary 1.6.9. *Let V be the vector space of global sections of $\mathbb{V}$, that is,*

$$V = \big\{ v \in \mathcal{C}^\infty(\mathcal{V}) \ : \ Dv = 0 \big\}.$$

Then σ leaves V invariant, and it induces a decomposition

$$V = V^+ \oplus V^-.$$

Proof. It is enough to prove that if $v \in V$, then also $v^+ \in V$; this further reduces to proving that $D(v^+) = 0$. Since $D(v) = 0$, we have $J(v) = \mathrm{d}^*\mathrm{d}v = 0$, and we can integrate by parts:

$$\begin{aligned} 0 &= \int_M \langle Jv, v \rangle \mathrm{d}\nu \\ &= \int_M \langle \mathrm{d}^{\mathrm{can}*}\mathrm{d}^{\mathrm{can}} v, v \rangle \mathrm{d}\nu + \sum_s \int_M \langle \tilde{\beta}(E_s) \cdot \big(\tilde{\beta}(E_s) \cdot v\big), v \rangle \\ &= \int_M \|\mathrm{d}^{\mathrm{can}} v\|_W^2 + \int_M \|\tilde{\beta} \cdot v\|^2, \end{aligned}$$

since $\beta(E_s)$ is self-adjoint (lemma 1.5.10). Hence both summands must vanish, and we obtain:

$$\mathrm{d}^{\mathrm{can}} v = 0; \quad \beta \cdot v = 0.$$

But then, since $\mathrm{d}^{\mathrm{can}}$ preserves the decomposition and β exchanges it,

$$\mathrm{d}^{\mathrm{can}}\big(v^+\big) = \big(\mathrm{d}^{\mathrm{can}} v\big)^+ = 0; \quad \beta \cdot \big(v^+\big) = \big(\beta \cdot v\big)^- = 0.$$

Hence also $\mathrm{d}(v^+) = \mathrm{d}^{\mathrm{can}}(v^+) + \beta \cdot v^+ = 0$. □

1.7 The case $M = S^1$

In this section we analyze briefly the special case where $M = S^1$. Although in many other sections we will require M to be a Kähler manifold, the case of S^1 has a special interest in his own, since the harmonic maps $f\colon \tilde{M} = \mathbb{R} \to N$ are the geodesics parametrized by a multiple of the arc length. We know that the symmetric spaces of non-compact type are complete and uniquely geodesic spaces, so given any two points y_0, y_1 in N we may find a unique geodesic arc connecting them. However, the existence of equivariant geodesics is more subtle.

In this case, since $\Gamma = \mathbb{Z}$, we have for the representation space $\mathbb{R}_{\mathrm{B}}(M,G) \overset{\mathrm{def}}{=} \mathrm{Hom}(\Gamma, G) = G$. The moduli space $\mathbb{M}_{\mathrm{B}}(\Gamma, G)$ identifies then with the GIT quotient of G modulo its action by conjugation, $\mathbb{M}_{\mathrm{B}}(M,G) = G/\!/G$. Every element here is represented by a semisimple element of G (in the case $G = \mathrm{SL}(n,\mathbb{C})$, this is a diagonalizable matrix). The energy defines a functional on $\mathbb{R}_{\mathrm{B}}(M,G)$ defined for $g = \rho(1) \in G$ by:

$$\mathcal{E}(g) = \inf\left\{\frac{1}{2}\int_0^1 \left\|\frac{\partial f(x)}{\partial x}\right\|^2 \mathrm{d}x \;:\; f\colon \mathbb{R} \to N, f(x+1) = g\cdot f(x)\right\}. \quad (1.13)$$

This infimum is a minimum if, and only if, a g-equivariant harmonic map exists. The energy functional induces one on $\mathbb{M}_{\mathrm{B}}(M,G)$, since if $g' = hgh^{-1}$ and f is g-equivariant, then $f' = h\cdot f$ is g'-equivariant, and the energy of these two functions coincide (as G acts by isometries on N). Remark that we can redefine the functional as:

$$\mathcal{E}(g) = \inf_{y\in N} \mathrm{dist}\big(y, g\cdot y\big)^2. \quad (1.14)$$

Indeed, on the one hand, for every curve $f\colon [0,1] \to N$ such that $f(0) = y$, $f(1) = g\cdot y$, as is every f in (1.13), we have $\frac{1}{2}\int_0^1 \left\|\frac{\partial f(x)}{\partial x}\right\|^2 \mathrm{d}x \geqslant \mathrm{dist}(y, g\cdot y)^2$, since the length of every curve is bigger than the distance of the two points, and using the Cauchy-Schwarz inequality. On the other hand, we can approximate this distance arbitarily well by smooth curves: Start with a geodesic arc connecting y to $g\cdot y$, parametrized by a multiple of the arc length, and juxtapose to it the translated by g. If this process gives a smooth curve, then it is an equivariant geodesic (but this is not always the case, see figure 1.2); conversely, by uniqueness of the geodesics, if an equivariant geodesic exists, it must be constructed in this way. If the curve is not smooth, we can approximate it arbitrarily well with equivariant and smooth curves, thus obtaining the equality of (1.13) and (1.14). In particular, the infimum in (1.14) is atteined if and only if the g-equivariant geodesic exists, that is, if and only if the Zariski closure of the 1-parameter subgroup generated by g is reductive. This is equivalent to g being semisimple, so that:

$$\min_{y \in N} \operatorname{dist}\big(y, g \cdot y\big) \text{ exists } \iff g \text{ is semisimple.} \tag{1.15}$$

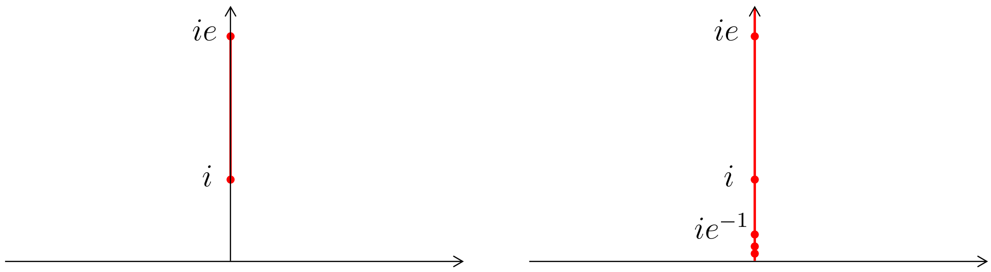

Figure 1.1: For $G = \mathrm{SL}(2, \mathbb{R})$ and $g = \begin{pmatrix} e^{\frac{1}{2}} & 0 \\ 0 & e^{-\frac{1}{2}} \end{pmatrix}$, taking a geodesic arc from i to $g \cdot i = ei$ and prolonging it gives the geodesic $f(t) = ie^t$.

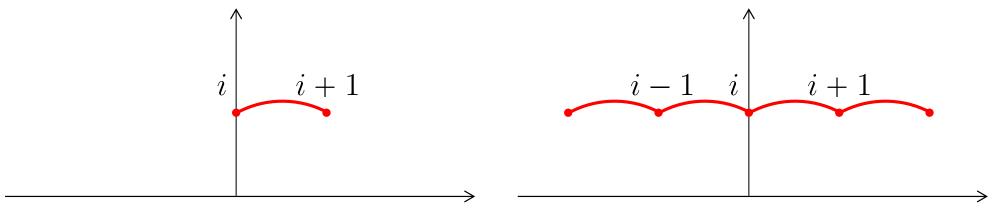

Figure 1.2: For $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, taking a geodesic arc from i to $g \cdot i = i + 1$ and prolonging it gives a non-smooth curve.

The square root of (1.14) is known as the translation length, and widely used for example in the study if CAT(0) spaces (cfr. [BH99], §II.6.1, where (1.15) is proved in this more general setting). For general finitely generated groups Γ instead of $\mathbb{Z}$, Parreau [Par11] proves a similar result for the analogous function

$$\inf_{y \in N} \sum_{s \in S} \operatorname{dist}(y, \rho(s) \cdot y)^2,$$

where S is a finite set of generators of Γ. That is, she proves that the infimum is achieved if and only if ρ is semisimple. By Corlette's theorem, the same is true for the energy functional.

1.8 Polarized harmonic local systems on Kähler manifolds

Complex polarized harmonic local systems enjoy particularly nice properties on Kähler manifolds. Let us start by pull-backs of phls.

Lemma 1.8.1. *Suppose that X' and X are compact Kähler manifolds, and $\varphi\colon X' \to X$ is holomorphic. Let $(\mathbb{V}, \sigma, S)$ be a (complex or real) phls on X. Then there is an induced pull-back (complex or real, respectively) phls $\varphi^*(\mathbb{V}, \sigma, S)$ on X'.*

Proof. The first four points of the definition pull-back with no problem in the more general setting of a smooth map between Riemannian manifolds. However, if $f\colon \tilde{X} \to \mathrm{GL}(n, \mathbb{C})/U(n)$ is harmonic, in general $\varphi \circ f$, which is the map giving the pull-back metric, will not be, even when φ is harmonic. But it does in our setting, as it follows from general results on (pluri)-harmonic maps and (pluri)-harmonic morphisms on Kähler manifolds. On the one hand, a map from a Kähler manifold to a symmetric space of non-compact type is harmonic if and only if it is pluriharmonic (that is, the restriction to every complex curve is harmonic). This follows from the Siu-Sampson Bochner's formula, cfr. [ABC+96], chapter VI. Then, thanks to [Lou99], we know that a holomorphic map between Kähler manifolds is a pluriharmonic morphism, that is, it pulls back pluriharmonic maps to pluriharmonic maps. □

Furthermore, since a phls gives us in particular a flat bundle with a harmonic metric, as in [Sim92] we also have the structure of a Higgs bundle:

Definition 1.8.2. A Higgs bundle $(\mathcal{E}, \theta)$ on a compact Kähler manifold X consists of a holomorphic vector bundle $(\mathcal{E}, \bar{\partial})$ and $\theta \in \mathcal{A}^1(\mathrm{End}\mathcal{E})$ a (1,0)-form such that

$$\bar{\partial}(\theta) = 0; \quad \theta \wedge \theta = 0.$$

Here, $\theta \wedge \theta$ is, up to a factor 2, what we wrote above as $[\theta, \theta]$, that is, we take wedge product on forms and the commutator of endomorphisms. We do not recall here the construction of the Higgs bundle associated to a phls, but we content ourselves to list how it fits in our setting (all of these properties are proved, or implicitly stated, in [Sim92]).

Proposition 1.8.3. *The Higgs bundle associated to a $\mathbb{C}$-phls $(\mathbb{V}, \sigma, S)$ has the following properties:*

1. *The smooth vector bundle underlying $\mathcal{E}$ is $\mathcal{V}$;*

2. *The 1-form* β *splits in* $(1,0)$ *and* $(0,1)$ *parts as*

$$\beta = \theta + \theta^*,$$

where θ^* *is the adjoint of* θ*, that is, for every two sections* v, w *of* $\mathcal{V}$ *and* χ *a (real) tangent vector,*

$$h\big(\theta(\chi) \cdot v, w\big) = h\big(v, \theta^*(\chi) \cdot w\big).$$

3. *The* $(0,1)$*-part of the canonical connection is the holomorphic structure* $\bar\partial$*.*

We can reformulate the definition of the energy of a map $f\colon \tilde{X} \to N$ in terms of the Higgs bundles. Indeed, the pull-back f^*h of the metric on the symmetric space (N,h) gives a symmetric 2-form whose $(1,1)$ part (which is closed, cfr. [Mok92], §1, Theorem 4) turns out to be $2 \cdot \mathrm{trace}(\theta \wedge \theta^*)$ (here the trace is taken on the endomorphism part). This follows easily from $f^*h = S(\beta,\beta)$, where S is the hermitian 2-form of the $\mathbb{C}$-phls, plus point 2. of proposition 1.8.3. Writing the energy density and the volume form in terms of a local orthonormal basis and denoting by Ω the Kähler form on X, then, it is immediate to show that

$$E(f) = \frac{1}{n!}\int_X \mathrm{trace}(\theta \wedge \theta^*) \wedge \Omega^{n-1}. \tag{1.16}$$

In particular, it follows from (1.16) that the energy of the harmonic map is independent of the metric on X in its Kähler class.

Remark 1.8.4. One can also prove directly that $\mathrm{trace}(\theta\wedge\theta^*)$ is closed: Indeed, recalling that $\mathrm{d} = \partial + \bar\partial + \mathrm{ad}(\theta) + \mathrm{ad}(\theta^*)$, the $(2,0)$ and $(0,2)$ terms of the Maurer-Cartan equation $\mathrm{d}^{\mathrm{can}}\beta = (\partial + \bar\partial)(\theta + \theta^*) = 0$, together with $\bar\partial(\theta) = \theta \wedge \theta = 0$, imply $\mathrm{d}\theta = [\theta^*, \theta] = \mathrm{d}(\theta^*)$. Thus:

$$\mathrm{d}\big(\mathrm{trace}(\theta \wedge \theta^*)\big) = \mathrm{trace}\big([\theta^*,\theta] \wedge \theta^* - \theta \wedge [\theta, \theta^*]\big).$$

Now each term vanishes independently thanks to the cyclic symmetry of the trace:

$$\mathrm{trace}\big([\theta^*,\theta] \wedge \theta\big) = \mathrm{trace}\big([\theta,\theta] \wedge \theta^*\big) = 0.$$

Of particular interest to us are the Kähler identities, which are proved in [Sim88], Lemma 3.1. Define the operators D' and D'' by

$$D' = \partial + \theta^*; \quad D'' = \bar\partial + \theta.$$

Fix a Kähler metric on X, so that we have an adjoint Lefschetz operator $\Lambda\colon \mathcal{A}^{p,q}(\mathcal{V}) \to \mathcal{A}^{p-1,q-1}(\mathcal{V})$. Then the usual Kähler identities hold for the dual operators D'^* and D''^* in terms of $[\Lambda, D'']$ and $[\Lambda, D']$. In particular, a form is harmonic if and only if it is $\Delta' = D'^*D' + D'D'^*$-harmonic (or equivalently, Δ''-harmonic). We will use the following consequence of this fact:

Lemma 1.8.5. *Let X be a Kähler manifold, $(\mathbb{V}, \sigma, S)$ a phls on it (or, more generally, $(\mathcal{V}, D, h)$ a harmonic bundle). Then a 1-form $\omega \in \mathcal{A}^1(\mathbb{V})$ is harmonic if and only if it is both D'- and D''-closed.*

Proof. The "only if" part is true in general, since if ω is harmonic then $\Delta'\omega = \Delta''\omega = 0$, and integrating by parts also $D'\omega = D''\omega = 0$. For the "if" part, we need ω to be a 1-form. Then, by the Kähler identities:

$$D'^*\omega = \frac{i}{2}[\Lambda, D'']\omega = \frac{i}{2}\Lambda D''\omega = 0.$$

This forces ω to be Δ'-harmonic, hence harmonic. □

Corollary 1.8.6. *Let $\varphi\colon X' \to X$ be a holomorphic map between Kähler manifolds, $(\mathbb{V}, \sigma, S)$ a phls on X and $\omega \in \mathcal{H}^1(X, \mathbb{V})$ a $\mathbb{V}$-valued harmonic 1-form. Then the pull-back $\varphi^*\omega$, which takes values in the pull-back phls, is harmonic, as well.*

Proof. The pull-back of a closed form is again closed. We prove that the same is true for D''-closeness, so that the result follows from the lemma. For the rest of the proof, every primed object, except possibly D' and D'', will live on X'. First notice that since the structure on X' is defined by pull-back, the 1-form β' and consequently the canonical connection $\mathrm{d}^{\mathrm{can}\prime}$ are the pull-back of β, $\mathrm{d}^{\mathrm{can}}$, respectively. Taking $(1,0)$ and $(0,1)$-parts, according to proposition 1.8.3, we have

$$\varphi^*\theta = \theta'; \quad \varphi^*\bar{\partial} = \bar{\partial}'.$$

Denoting by $\omega' = \varphi^*\omega$, we have

$$D''\omega' = D''\varphi^*\omega = \varphi^*D''\omega = 0.$$

□

We conclude this section with what could be thought as an infinitesimal correspondence between the moduli space of flat connections and that of Higgs bundles (which, however, holds true also in the Kähler, non-projective, setting). Let $(\mathcal{V}, D)$ be a flat bundle and $(\mathcal{E}, D'')$ a Higgs bundle on X, corresponding through a ρ-equivariant, harmonic metric $f\colon \tilde{X} \to N$.

Definition 1.8.7. A first order deformation of D is a closed 1-form $\omega \in Z^1(X, \mathrm{End}(\mathcal{V}))$. A first order deformation of D'' is a Dolbeault 1-cocycle $B \in Z^1_{\mathrm{Dol}}(X, \mathrm{End}(\mathcal{E}))$.

The reason for this terminology is evident: On the one hand, $\mathrm{d} + t\omega$ is a flat derivation; on the other hand, $D'' + tB$ is an operator of the form needed to give rise to a Higgs bundle. Simpson proves a formality result, which for our needs will be:

Lemma 1.8.8 ([Sim92], pag 24)**.** *There is a natural isomorphism*

$$H^1_{DR}(X, \mathrm{End}(\mathcal{V})) \cong H^1_{\mathrm{Dol}}(X, \mathrm{End}(\mathcal{E})) \cong H^1\big(\mathcal{A}^\bullet(\mathcal{E}), D''\big).$$

The first two cohomology spaces are indeed isomorphic to the space of harmonic 1-forms, which embeds in the spaces of 1-cocycles. Hence, given a first order deformation of a flat connection, we can construct (up to a coboundary) a first order deformation of a Higgs bundle, and conversely.

1.9 Complex Variations of Hodge structure

We recall the definition of complex variations of Hodge structure ($\mathbb{C}$-VHS for short), which provide a crucial class of polarized harmonic local systems on a compact Kähler manifold X.

Definition 1.9.1. A complex variation of Hodge structure ($\mathbb{C}$-VHS for short) on X is a $\mathcal{C}^\infty$ flat complex vector bundle $(\mathcal{V}, D) \to X$ with a Hodge decomposition

$$\mathcal{V} = \bigoplus_{p+q=w} \mathcal{V}^{p,q}$$

such that the flat connection satisfies the Griffiths transversality condition:

$$D\colon \mathcal{V}^{p,q} \to \mathcal{A}^{0,1}(V^{p+1,q-1}) \oplus \mathcal{A}^{1,0}(V^{p,q}) \oplus \mathcal{A}^{0,1}(V^{p,q}) \oplus \mathcal{A}^{1,0}(V^{p-1,q+1}). \quad (1.17)$$

We require further that the structure be polarized, that is, that there exists a flat hermitian form S which makes the Hodge decomposition orthogonal and that is positive definite on $\mathcal{V}^{p,q}$ for p even and negative definite on $\mathcal{V}^{p,q}$ for p odd. The integer w is called the *weight* of the $\mathbb{C}$-VHS.

We denote the Hodge filtration by

$$\mathcal{F}^p = \bigoplus_{s \geqslant p} \mathcal{V}^{s,w-s}.$$

Lemma 1.9.2. *Let X be a compact Kähler manifold, and let $(\mathcal{V}, D, S)$ be a $\mathbb{C}$-VHS on it. Then there is a corresponding $\mathbb{C}$-phls structure on the same bundle.*

Proof. The flat bundle and local system structures are the same, as is the polarization S. The involution σ is clearly defined as $(-1)^p$ on $\mathcal{V}^{p,q}$. The only non-trivial part is the harmonicity of the metric, but this is classic. We will review the construction, without detailing the proof.

Fix a point $x_0 \in X$, we write $F^p \supseteq F^{p+1} \supseteq \ldots$ for the flag of subspaces of $\mathcal{V}_{x_0}$ induced by $\mathcal{F}^\bullet$. Let G_0 inside $\mathrm{GL}(\mathcal{V}_{x_0})$ be the orthogonal group of S, and V_0 the compact subgroup fixing the flag $\{F^\bullet\}$. Then $D = G_0/V_0$ is called the period domain and the $\mathbb{C}$-VHS determines a *period mapping*

$$\Phi\colon \tilde{X} \to D = G_0/V_0$$

which is holomorphic. Let K_0 be the maximal compact subgroup containing V_0. Composing Φ with the projection we obtain a map

$$f\colon \tilde{X} \xrightarrow{\Phi} D \xrightarrow{\pi} N = G_0/K_0 \to \mathrm{GL}(\mathcal{V}_{x_0})/K$$

which is harmonic. This is the harmonic metric of the phls. □

Remark 1.9.3. Thanks to [Sim92], we can work in a "smaller" setting: If we consider G the complex Zariski closure of the image of the monodromy ρ_0, and let G_0 be the subgroup preserving S, then we obtain another, smaller period domain of the form G_0/V_0. In fact, in this case G_0 coincides with the real Zariski closure of $\rho_0(\Gamma)$, so it is a group of Hodge type (in particular, reductive) and a real form for G.

As we did in section 1.8, since we have a phls structure on a Kähler manifold, there is an induced structure of Higgs bundle $(\mathcal{E}, \theta)$. Simpson calls the Higgs bundle thus obtained a "system of Hodge bundles" (cfr. [Sim88]):

Definition 1.9.4. A system of Hodge bundles is a Higgs bundle $(\mathcal{E}, \theta)$ with a decomposition of locally free sheaves $\mathcal{E} = \bigoplus \mathcal{E}^{r,s}$ such that

$$\theta\colon \mathcal{E}^{r,s} \to \mathcal{E}^{r-1,s+1} \otimes \Omega^1_X.$$

Remark 1.9.5. Since we know that $\mathrm{d} = \mathrm{d}^{\mathrm{can}} + \beta = \partial + \bar{\partial} + \theta + \theta^*$, the transversality condition (1.17) means exactly that

$$\partial\colon \mathcal{V}^{p,q} \to \mathcal{A}^{1,0}(\mathcal{V}^{p,q}), \quad \theta\colon \mathcal{V}^{p,q} \to \mathcal{A}^{1,0}(\mathcal{V}^{p-1,q+1}),$$

and similarly for $\bar{\partial}$ and θ^*.

There is another characterization of Higgs bundles which are induced by a $\mathbb{C}$-VHS, in terms of the $\mathbb{C}^*$-action on the space of Higgs bundles:

$$\forall t \in \mathbb{C}^*, \quad t \cdot (\mathcal{E}, \theta) = (\mathcal{E}, t\theta). \tag{1.18}$$

One can check immediately that for every t, the pair $(\mathcal{E}, t\theta)$ is again a Higgs bundle. Then, Simpson proves:

Theorem 1.9.6 ([Sim92], Lemma 4.1)**.** *A Higgs bundle $(\mathcal{E}, \theta)$ is induced by a $\mathbb{C}$-VHS if and only if there is an isomorphism of Higgs bundles*

$$(\mathcal{E}, t\theta) \cong (\mathcal{E}, \theta)$$

for some (hence, for every) t which is not a root of 1.

Chapter 2

The universal twisted harmonic map

Introduction au chapitre

Le but de ce chapitre est de construire une application harmonique équivariante universelle $\mathscr{H}$, d'en démontrer la continuité et d'utiliser ce dernier fait pour déduire la continuité de la fonctionnelle de l'énergie sur l'espace des représentations $\mathrm{Hom}(\Gamma, G)$.

Par la non-unicité dans le théorème de Corlette, pour obtenir une application universelle il faut fixer un paramètre supplémentaire. En considérant un point $\tilde{x}_0 \in \tilde{M}$, soit Y les sous-ensemble des (n, ρ) dans $N \times \mathrm{Hom}(\Gamma, G)$ tel qu'il existe une application harmonique, ρ-équivariante et telle que $f(\tilde{x}_0) = n$. On a alors une application bien définie $\mathscr{H} : Y \times \tilde{M} \to N$. La continuité de cette application est démontrée dans la Proposition 2.2.1; cela suit des faits que l'énergie est localement bornée sur $\mathrm{Hom}(\Gamma, G)$, de l'estimation du module de Lipschitz d'une application harmonique qui ne dépend que de son énergie (voir par exemple [Lin99]) et des itérations usuelles sur l'équation d'harmonicité (qui est semi-linéaire et elliptique).

La continuité de $\mathscr{H}$ entraine sans problèmes la continuité de la fonctionnelle de l'énergie sur l'espace des modules des représentations, qui est le quotient des représentations semi-simples par l'action de G par conjugaison. En démontrant que l'énergie d'une représentation est égale à l'énergie de sa semi-simplifiée (c'est-à-dire, la fonctionnelle de l'énergie est constante sur l'adhérence des orbites), et grâce à un argument utilisant la propreté du quotient de Kempf–Ness, on arrive à démontrer la continuité de la fonctionnelle de l'énergie sur l'entièreté de $\mathrm{Hom}(\Gamma, G)$.

Il faut noter que tous ces résultats sont évidents si on ne considère que des représentations Zariski-dense: en effet, dans ce cas on peut construire les applications harmoniques en famille, et l'application $\mathscr{H}$ est même lisse, voir [Cor91] (toute représentation Zariski-dense est automatiquement semi-simple).

2.1 Upper semi-continuity of the energy

We start by fixing a base point $x_0 \in M$ in order to construct a universal harmonic and equivariant mapping over the semisimple locus $\mathbb{R}_{\mathrm{B}}(M, x_0, G)^{ss}$.

Definition 2.1.1. Fix $\tilde{x}_0$ a preimage of x_0. We will denote by Y the subset of $N \times \mathbb{R}_{\mathrm{B}}(M, x_0, G)$ given by the points (n, ρ) such that there exists a ρ-equivariant harmonic map f verifying $f(\tilde{x}_0) = n$.

The projection of Y on the second component, $\pi_2(Y)$, gives exactly the semisimple representations, by Corlette's theorem 1.1.1. Tautologically, we have a universal harmonic mapping $\mathscr{H} : Y \times \tilde{M} \to N$ defined by $\mathscr{H}(n, \rho, \tilde{x}) = f(\tilde{x})$, where f is the unique harmonic ρ-equivariant map such that $f(\tilde{x}_0) = n$. Recall that the energy functional on $\mathbb{R}_{\mathrm{B}}(M, G)$ is defined by:

$$E(\rho) = \inf_{\substack{f \in \mathcal{C}^\infty(\tilde{M}; N) \\ \rho-\text{equivariant}}} E(f).$$

Then, since harmonic maps to a symmetric space of non-compact type are energy-minimizing, the restriction to $\mathbb{R}_{\mathrm{B}}(M, G)^{ss}$ is given by

$$E(\rho) = E\big(\mathscr{H}(\rho, n, \cdot)\big),$$

where $n \in N$ is such that $(n, \rho) \in Y$. In particular, continuity or smoothness of E will follow from the corresponding properties of $\mathscr{H}$.

Lemma 2.1.2. *Let $\rho_t : \Gamma \to G$ be a smooth family of representations, for t in some smooth parameter space T, and $f : \tilde{M} \to N$ a ρ_0-equivariant map. Then we can always find a smooth family $f_t : \tilde{M} \to N$ of ρ_t-equivariant maps, such that $f_0 = f$. In particular, the energy functional is upper semi-continuous on all of $\mathbb{R}_{\mathrm{B}}(M, G)$.*

Proof. As usual, smooth ρ_t-equivariant maps correspond to metrics on $\mathcal{V}_t = \tilde{M} \times_\Gamma \mathbb{C}^n$, which is a smooth family of vector bundles over M. All those bundles are trivialized over common open subsets (namely, sufficiently small open subsets U of M which are well-covered by $\tilde{M} \to M$), with a flat local frame:

$$\begin{aligned} s_U^t : U \times \mathbb{C}^n &\to \pi_t^{-1} U \\ (x, \mathbf{e}) &\mapsto [\tilde{x}, \mathbf{e}], \end{aligned}$$

where $\pi_t\colon \mathcal{V}_t \to M$ and $\tilde{x} \in \tilde{U}$ is a preimage of x (the construction depends on the choice of connected preimages $\tilde{U}$ for each U, independent of x). The map f corresponds to a metric h^0 on $\mathcal{V}_0$, and in turn to a family of metrics on $\mathbb{C}^n$, one for every pair (x, U) with $x \in U$, given by

$$\big\langle \mathbf{e}, \mathbf{f} \big\rangle_{x,U} = h_x\big(s^0_U(x, \mathbf{e}), s^0_U(x, \mathbf{f})\big), \quad \mathbf{e}, \mathbf{f} \in \mathbb{C}^n.$$

The compatibility condition reads $\langle \mathbf{e}, \mathbf{f}\rangle_{x,U} = \langle \mathbf{e}, \mathbf{f}\rangle_{x,V}$ every time $x \in U \cap V$. Let $\{\chi_U\}$ be a partition of the unity subordinated to the open cover $\{U\}$, and define the family of metrics by

$$h^t_x\big(v_t, w_t\big) = \sum_{U \ni x} \chi_U(x) \big\langle (s^t_U)^{-1}(v_t), (s^t_U)^{-1}(w_t)\big\rangle.$$

This is smooth in t and extends the metric h^0, as wanted.

Semi-continuity of the energy follows easily: Let $\rho_0\colon \Gamma \to G$ be any representation, and $f^n\colon \tilde{M} \to N$ be an energy-minimizing sequence of ρ_0-equivariant maps. Given any converging family of representations $\rho_t \to \rho_0$, we deform each f^n to an f^n_t as above. Since f^n_t is a smooth family, f^n_t converges to f^n in $L^{1,2}_{\mathrm{loc}}$, so that in particular $E(f^n_t)$ converges to $E(f^n)$ (since the integral defining the energy is taken on a compact fundamental domain). Hence

$$\mathcal{E}(\rho_0) = \lim_n E(f^n) \geqslant \lim_n E(f^n_t) - \varepsilon(t) \geqslant \mathcal{E}(\rho_t) - \varepsilon(t),$$

for some infinitesimal function $\varepsilon(t)$. Passing to the limit in t, we obtain $\mathcal{E}(\rho_0) \geqslant \limsup_t \mathcal{E}(\rho_t)$. □

Remark that, in particular, the energy functional is locally bounded.

2.2 Continuity of the universal harmonic map

In this section, thanks to a Lipschitz estimate due to Lin [Lin99], we shall prove the continuity of $\mathscr{H}$ and the closure of Y.

Proposition 2.2.1. *The subset $Y \subset N \times \mathbb{R}_{\mathrm{B}}(M, x_0, G)$ is closed. The universal harmonic mapping $\mathscr{H}\colon Y \times \tilde{M} \to N$ is continuous.*

Proof. Take a converging sequence $(n_t, \rho_t) \to (n_\infty, \rho_\infty)$ of points in Y, and let f_t be the harmonic ρ_t-equivariant mapping such that $f_t(\tilde{x}_0) = n_t$. We want to prove that f_t converges in $\mathcal{C}^0$ and in $W^{2,p}$ to some function, which will necessarily be the harmonic ρ_∞-equivariant map f_∞ such that $f_\infty(\tilde{x}_0) = n_\infty$. This fact will easily imply both statements.

By lemma 2.1.2, the energy of the family f_t is bounded. We can apply Theorem A of [Lin99]: This (or rather, its proof, see Section 5, loc. cit.) implies that inside a sufficiently small geodesic ball $B(\tilde{x}, R)$ in $\tilde{M}$ the sup norm of $\mathrm{d}f_t$ is bounded by a constant which depends only on the geodesic ball, on the geometries of $\tilde{M}$ and N and on the energy $E(f_t)$. In particular, restricting ourselves to any compact subset $K \ni \tilde{x}_0$ of $\tilde{M}$, the family $f_t\big|_K$ consists of Lipschitz maps, with a uniform Lipschitz constant L. This, together with the condition $f_t(\tilde{x}_0) \to n_\infty$, implies that they are also uniformly bounded. Then, we can apply the $W^{2,p}$-estimates (cfr. [GT77], theorem 9.11) to the semi-linear second order elliptic equation of the harmonic maps, which in local coordinates is:

$$0 = \tau(f)^\alpha = -\Delta f^\alpha + {}^N\Gamma^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j g^{ij}. \tag{2.1}$$

This gives a uniform bound on the $W^{2,p}$-norm of f_t and, in turn, a $W^{1,p}$ uniform bound on the right hand side of (2.1). Iterating the argument we obtain uniform bounds in every $W^{k,p}$; in particular, thanks to the Sobolev embedding theorem and Arzelà–Ascoli theorem, we can extract a subsequence converging in $\mathcal{C}^2$ to some smooth map f. This is automatically ρ_∞-equivariant and satisfies $f(\tilde{x}_0) = n_\infty$, and it is harmonic being a strong solution to (2.1). This proves that Y is closed.

To obtain the continuity of $\mathscr{H}$, we only need to remark that in fact the whole sequence f_t converges to f_∞: Taking any subsequence $f_{t'}$ and applying the argument above, we find a converging sub-subsequence $f_{t''}$; the uniqueness of f_∞ allows us to conclude that $\lim f_{t''} = f_\infty$. This forces $f_t \to f_\infty$; then, an easy argument using uniform convergence and the triangular inequality gives the continuity of $\mathscr{H}$. □

Recall that, given any representation $\rho\colon \Gamma \to G$, there is an associated *semisimplification*, defined as the graded associated to any composition series. Considering the action of G by conjugation on the set of representations, we will call the semisimplification of ρ any point of the unique closed orbit included in the closure of the orbit of ρ. This is defined only up to conjugation, but that will be of no consequence in our setting. In the following, denote by ρ^{ss} such a semisimplification of ρ.

Lemma 2.2.2. *Let $\rho\colon \Gamma \to G$ be any representation. Then $E(\rho) = E(\rho^{ss})$.*

Proof. First remark that the proof of Corlette's theorem in [Cor88] allows us to construct a sequence of ρ-equivariant maps f_n such that $E(f_n)$ converges to $E(\rho)$ and also such that L^p norms of the first two derivatives of f_n are bounded. Indeed, in §4, Corlette proves that, given any flat connection D_0

(for us: any metric $\tilde{f}_0$) the heat flow starting from D_0 is defined for every time, thus constructing D_t (hence, $\tilde{f}_t$). Furthermore, the energy is decreasing along the flow, and, denoting by Φ_t his moment map for D_t (which is just the tension field of $\tilde{f}_t$), we have $\|\Phi\|_{L^\infty} \to 0$. This follows since, in his notations,

$$\left\|\Phi_t\right\|_{L^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left\|\theta_t\right\|_{L^2} \to 0 \quad \text{and} \quad \frac{\partial}{\partial t}\left|\Phi_t\right| + \Delta\left|\Phi_t\right| \leqslant 0.$$

The greatest part of his proof is devoted to giving an estimate on the $W^{1,p}$-norm of $\theta_t = \mathrm{d}\tilde{f}_t$ depending only on the L^2-norm of θ_t (whose square is the energy) and the L^∞ norm of Φ_t (the other constants depending only on the geometry of M and N). Observe that, since the latter converges to 0, for t big enough such $W^{1,p}$ norms ultimately depend only on $\|\theta_t\|_{L^2}$, hence in turn on $\|\theta_0\|_{L^2}$ (since this norm is decreasing along the flow).

Now take any minimizing sequence g^n for ρ, i.e., a sequence of ρ-equivariant maps such that $E(g^n) \to E(\rho)$ in a monotone way. For each n, start the heat flow from g^n; we obtain a family g^n_t with the properties above. By what we have just said, we can find for t big enough a g^n_t such that the $W^{1,p}$ norms of $\mathrm{d}g^n_t$ are bounded by a constant depending only on the geometries of M and N and on $E(g^n) \leqslant E(g^0)$. Define $f_n = g^n_t$, for such a choice of t; in this way f_n has bounded energy and also the first and second derivatives have bounded L^p norms.

Now suppose that ρ is not already semi-simple and fix a point $\tilde{x}_0 \in \tilde{M}$. Clearly, in this case, the convergence of $f_n(\tilde{x}_0)$ must fail, otherwise, thanks to the derivative estimates, we could construct a limit map as in the preceeding proposition, which would be a harmonic ρ-equivariant mapping. So let $g_n \in G$ be such that $f_n(\tilde{x}_0) = g_n K$. In this way, $g_n^{-1} f_n(\tilde{x}_0) = eK$ and so an adequate subsequence of $\tilde{f}_n = g_n^{-1} f_n$ converges up to the first derivatives to some map $\tilde{f}_\infty$, which a priori is just $\mathcal{C}^1$. Define $\tilde{\rho}_n = g_n^{-1} \cdot \rho \cdot g_n$, so that $\tilde{f}_n$ is $\tilde{\rho}_n$-equivariant. We claim that $\tilde{\rho}_n$ converges to some $\tilde{\rho}_\infty$. Indeed, for every $\gamma \in \Gamma$, the Lipschitz estimate applied to $\tilde{f}_n$ implies:

$$\operatorname{dist}\big(eK, \tilde{\rho}_n(\gamma)K\big) = \operatorname{dist}\big(\tilde{f}_n(\tilde{x}_0), \tilde{f}_n(\gamma\tilde{x}_0)\big) \leqslant L \cdot \operatorname{dist}(\tilde{x}_0, \gamma\tilde{x}_0) = C(\gamma)$$

(in fact here the Lipschitz constant L depends on γ, as well). Denoting by $\pi\colon G \to G/K$ the projection $g \mapsto gK$, we have that $\tilde{\rho}_n(\gamma) \in \pi^{-1}(\overline{B(eK, C(\gamma))})$, which is compact, hence there is a subsequence $\tilde{\rho}_{n'}(\gamma)$ converging to some $\tilde{\rho}_\infty(\gamma)$. Repeating the procedure over a finite set of generators we find the desired limit representation. In this way, we have constructed a $\tilde{\rho}_\infty$-equivariant map $\tilde{f}_\infty$, which is $\mathcal{C}^1$ and in fact must be harmonic since it is a minimizer of the energy. To see this, remark that $E(\tilde{f}_n) = E(f_n)$ converges on the one hand to $E(\rho)$, by construction of f_n, and on the other hand to $E(\tilde{f}_\infty)$, by convergence in $W^{1,2}$. Then, making use of the semi-continuity of lemma 2.1.2, we get

$$E(\tilde{f}_\infty) \geqslant E(\tilde{\rho}_\infty) \geqslant \limsup E(\tilde{\rho}_n) = E(\rho) = \lim E(f_n) = E(\tilde{f}_\infty).$$

Hence $E(\tilde{f}_\infty) = E(\tilde{\rho}_\infty)$, as claimed, and also $E(\rho) = E(\tilde{\rho}_\infty)$; but as $\tilde{\rho}_\infty$ admits a equivariant harmonic mapping, hence it is semisimple, and since it belongs to the closure of the orbit of ρ, we have $\tilde{\rho}_\infty = \rho^{ss}$, which concludes the proof. □

Proposition 2.2.3. *The energy functional is continuous on the whole of* $\mathbb{R}_{\mathrm{B}}(M, G)$.

Proof. We start by proving the proposition on the subspace $\mathbb{R}_{\mathrm{B}}(M, G)^{ss}$. So consider a converging sequence of semisimple representations $\rho_t \to \rho_\infty$, and suppose that ρ_∞ is semisimple as well. First suppose that there exists a sequence $n_t \in N$ such that $(n_t, \rho_t) \in Y$ and that n_t converges to some $n_\infty \in N$ as t tends to ∞. Then, by closedness of Y, $(n_\infty, \rho_\infty) \in Y$, hence we can apply the $W^{1,2}$ convergence in the proof of the proposition to get

$$E(\rho_t) = E\big(\mathscr{H}(n_t, \rho_t, \cdot)\big) \xrightarrow{t\to\infty} E\big(\mathscr{H}(n_\infty, \rho_\infty, \cdot)\big) = E(\rho_\infty).$$

If such a sequence does not exist, choose some n_∞ such that $(n_\infty, \rho_\infty) \in Y$. Then there exist $g_t \in G$ such that

$$(n_\infty, \tilde{\rho}_t) \in Y, \qquad \tilde{\rho}_t = g_t \cdot \rho_t \cdot g_t^{-1}.$$

Proceeding as in the proof of the lemma, one obtains that $\tilde{\rho}_t$ converges to some $\tilde{\rho}_\infty$. Again, this is semisimple (in this case it follows directly from proposition 2.2.1, since $(n_\infty, \tilde{\rho}_\infty)$ must be in Y). Hence, $\tilde{\rho}_\infty$ is in fact conjugate to ρ_∞, since the moduli space $\mathbb{M}_{\mathrm{B}}(M, G)$ is Hausdorff if one only considers semisimple (i.e. closed) points. Then, using what we already proved in the first part with $n_t \equiv n_\infty$, we conclude that:

$$E(\rho_t) = E(\tilde{\rho}_t) \xrightarrow{t\to\infty} E(\tilde{\rho}_\infty) = E(\rho_\infty).$$

We now proceed to the general case. After taking a faithful linear representation of G inside $\mathrm{GL}(n, \mathbb{C})$, which gives an immersion $\mathbb{R}_{\mathrm{B}}(M, G) \subseteq \mathbb{R}_{\mathrm{B}}(M, \mathrm{GL}(n, \mathbb{C}))$, we only need to prove the statement for the case $G = \mathrm{GL}(n, \mathbb{C})$, since the energy of a representation is independent of totally geodesic embeddings $G/K \subset \mathrm{GL}(n, \mathbb{C})/U(n)$. In fact, we will only need to suppose G to be a complex reductive algebraic group, which we do from now on (K will still denote one of its maximal subgroups). Consider a converging sequence of representations $\rho_n \to \rho_\infty$, call ρ_n^{ss}, ρ_∞^{ss} the respective semisimplifications, and denote by $\{\rho_n^{ss}\}$, $\{\rho_\infty^{ss}\}$ the closed points of $\mathbb{M}_{\mathrm{B}}(M, G)$ they represent. Since the functions on $\mathbb{M}_{\mathrm{B}}(M, G)$ are generated by the traces, and $\mathrm{trace}(\rho_n) = \mathrm{trace}(\rho_n^{ss})$ converges to $\mathrm{trace}(\rho_\infty) = \mathrm{trace}(\rho_\infty^{ss})$, we have convergence of the closed points in $\mathbb{M}_{\mathrm{B}}(M, G)$. Then, the homeomorphism given by the Kempf-Ness theorem reads

$$\mathbb{M}_{\mathrm{B}}(M, G) \cong \mu^{-1}(0)/K, \quad \mu^{-1}(0) \subseteq \mathbb{R}_{\mathrm{B}}(M, G)^{ss},$$

where μ is a moment map and the quotient is meant in the usual sense. Then, the closed points $\{\rho_n^{ss}\}$ can be lifted to some $\tilde{\rho}_n^{ss}$ in $\mu^{-1}(0)$, which, by hypothesis, are conjugated to ρ_n^{ss}. By properness of the projection $\mu^{-1}(0) \to \mu^{-1}(0)/K$, a subsequence of this converges to some $\tilde{\rho}_\infty^{ss} \in \mu^{-1}(0)$. Its class in $\mathbb{M}_{\mathrm{B}}(M, G)$ must be that of ρ_∞^{ss}, hence ρ_∞^{ss} is conjugated to $\tilde{\rho}_\infty^{ss}$. Applying what we already proved on semisimple representations together with lemma 2.2.2, we conclude the proof:

$$E(\rho_n) = E(\rho_n^{ss}) = E(\tilde{\rho}_n^{ss}) \xrightarrow{n \to \infty} E(\tilde{\rho}_\infty^{ss}) = E(\rho_\infty^{ss}) = E(\rho_\infty).$$

□

Remark 2.2.4. If we consider only Zariski-dense representations, all the results of this section are trivial. Indeed, in that case, thanks to the uniqueness of the harmonic representative, one can directly construct the harmonic maps in families, thus obtaining a smooth family of harmonic f_t's. A direct check of the proof of Corlette's theorem shows that one can consider a family of flat vector bundles $\mathcal{V}_t$, for small t, induced by a smooth family of representations ρ_t; then, the metric is constructed on all bundles simultaneously, and smoothness can be proved. The details are however long to check; more easily, one can apply an implicit function theorem to use Corlette's result directly. This is done in [Cor91], Proposition 2.3.

Chapter 3

First order harmonic maps

Introduction au chapitre

Dans ce chapitre on va fixer une représentation semi-simple $\rho_0\colon \Gamma \to G$ et une application harmonique ρ_0-équivariante $f\colon \tilde{M} \to N$. On se propose de décrire toute déformation v de f qui est à la fois harmonique et équivariante le long d'une déformation au premier ordre $\rho_t^{(1)}$ de ρ_0. On commence par définir les déformations au premier ordre $\rho_t^{(1)}$ et v, modelées sur les dérivées premières $\frac{\partial \rho_t}{\partial t}\big|_{t=0}$ et $\frac{\partial f_t}{\partial t}\big|_{t=0}$, respectivement; ensuite, on dit qu'une déformation v de f est harmonique (respectivement: $\rho_t^{(1)}$-équivariante) si elle satisfait la même condition au premier ordre que la dérivée d'une famille d'applications harmoniques (respectivement: ρ_t-équivariantes). Le point crucial pour ce qui suit est qu'une déformation harmonique est le zéro d'un opérateur (l'opérateur de Jacobi) formellement identique à l'expression trouvée pour le laplacien Δ dans le chapitre 1.

Le théorème principal du chapitre est un résultat permettant de construire toute déformations harmoniques et $\rho_t^{(1)}$-équivariantes comme suit: la déformation $\rho_t^{(1)}$ correspond à un 1-cocycle de groupes $c \in Z^1(\Gamma, \mathfrak{g})$. Sa classe de cohomologie peut être représentée, grâce au théorème de Hodge, par une 1-forme harmonique ω à valeurs dans le système local adjoint $\mathrm{Ad}(\rho_0)$. N'importe quelle primitive $F\colon \tilde{M} \to \mathfrak{g}$ de cette 1-forme vérifie l'équation $\Delta(F) = 0$, en donnant donc un candidat pour une déformation harmonique. En effet, on arrive à démontrer qu'on peut toujours imposer de plus la bonne équivariance à F, d'une façon à ce que la projection $\vartheta_{TN}(f, F)$ soit une déformation v harmonique et $\rho_t^{(1)}$-équivariante. De plus, toutes telles déformations se construisent de cette façon car l'application ϑ_{TN} est affine et surjective de l'espace des primitives F, qui est un $\mathfrak{h}$-torseur, $\mathfrak{h}$ étant l'algèbre de Lie du centralizateur $H = Z_G(\mathrm{Image}(\rho_0))$ sur l'espace des déformations harmoniques $\rho_t^{(1)}$-équivariantes, qui est un $\mathfrak{h} \cap [\mathfrak{p}]$-torseur. En particulier, toute application harmonique est déformable au premier ordre, le long de n'importe quelle déformation de ρ_0.

Le chapitre se termine avec un résultat de fonctorialité dans le cas où $M = X$ est une variété kählerienne suivi d'une déscription élementaire du cas abelien, c'est-à-dire, en supposant $G = \mathbb{C}^*$.

3.1 First order deformations

Definition 3.1.1. A *first order deformation* of a smooth map $f\colon \tilde{M} \to N$ of a manifold $\tilde{M}$ into another such manifold N is a smooth map $v\colon \tilde{M} \to TN$ such that $\pi_N \circ v = f$.

Slightly abusing notation, we will always see v as a section of f^*TN, and sometimes write $v = \left(f, \frac{\partial f_t}{\partial t}\big|_{t=0}\right)$. For the greatest generality, and motivated by the algebraic case, where $TN = \mathbf{N}(\mathbb{R}[t]/(t^2))^\circ$ (see remark 1.4.4 and the conditions therein), we will just work with t a formal parameter such that $t^2 = 0$.

If $v\colon \tilde{M} \to TN$ is a first order deformation, a lift of v is a smooth map $F\colon \tilde{M} \to \mathfrak{g}$ such that $\vartheta_{TN} \circ (f, F) = v$. Such lifts exist (for example take $F = \beta_N(v)$), but they are not unique, since the $[\mathfrak{k}]$ part is undetermined: In general, for every lift F, we may write $F = \beta_N(v) + \kappa(x)$, where $\kappa \in \mathcal{C}^\infty(\tilde{M}, [\mathfrak{k}])$, i.e. $F^{[\mathfrak{p}]} = \beta_N(v)$.

Definition 3.1.2. Let $\rho_0\colon \Gamma \to G$ be a representation. A *first order deformation* of ρ_0 is a representation $\rho_t^{(1)}\colon \Gamma \to TG$ that lifts ρ_0 (here t is meant as a formal parameter such that $t^2 = 0$). The associated 1-cocycle c is defined as

$$c(\gamma) = \frac{\mathrm{d}\rho_t^{(1)}(\gamma)}{\mathrm{d}t}\Big|_{t=0} \cdot \rho(\gamma)^{-1} \in \mathfrak{g}. \tag{3.1}$$

Indeed, it is easy to see that the data of a first order deformation of a representation is equivalent to that of a representation ρ_0 together with a 1-cocycle c for the adjoint action, that is, a map $c\colon \Gamma \to \mathfrak{g}$ such that

$$c(\gamma\eta) = c(\gamma) + \mathrm{Ad}_{\rho_0(\gamma)}(\eta).$$

3.2 Equivariant deformations

Definition 3.2.1. Let $\rho_t^{(1)} = (\rho_0, c)\colon \Gamma \to TG$ be a first order deformation of $\rho_0\colon \Gamma \to G$, and $f\colon \tilde{M} \to N$ a ρ_0-equivariant map. A first-order deformation

v of f is $\rho_t^{(1)}$-equivariant if $v(\gamma\tilde{x}) = \rho_t^{(1)}(\gamma)\cdot v(\tilde{x})$, where $\cdot$ stands for the action of TG on TN. Explicitely, using the description of the action of section 1.4, this is equivalent to:

$$v(\tilde{x}) - \rho(\gamma)_* v(\gamma^{-1}\tilde{x}) = \vartheta_{TN}\big(f(\tilde{x}), c(\gamma)\big),$$

where ϑ_{TN} is defined as in (1.5)

Example 3.2.2. Of course, the main example of a $\rho_t^{(1)}$-equivariant first order deformation v of f is obtained considering smooth families of representations $\rho_t\colon \Gamma \to G$, for $t \in (-\varepsilon, \varepsilon)$ and of ρ_t-equivariant maps $f_t\colon \tilde{M} \to N$. Then, a straightforward computation proves that, defining

$$c(\gamma) = \frac{\partial \rho_t(\gamma)}{\partial t}\Big|_{t=0} \cdot \rho_0(\gamma)^{-1}, \quad v(\tilde{x}) = \frac{\partial f_t(\tilde{x})}{\partial t}\Big|_{t=0},$$

we obtain a first order $\rho_t^{(1)}$-equivariant deformation v of f.

Definition 3.2.3. Let $\rho_t^{(1)}$ be a first-order deformation of a representation $\rho_0\colon \Gamma \to G$. A function $F\colon \tilde{M} \to \mathfrak{g}$ is $\rho_t^{(1)}$-equivariant if

$$F(\gamma\tilde{x}) = \mathrm{Ad}_{\rho_0(\gamma)} F(\tilde{x}) + c(\gamma).$$

Lemma 3.2.4. *A map $(f, F)\colon \tilde{M} \to N \times \mathfrak{g}$ is $\rho_t^{(1)}$-equivariant in the sense above if and only if it is for the action of TG on $N \times \mathfrak{g}$ given by diagram* (1.7)*. Hence, if (f, F) is equivariant, then the first order deformation of f defined by*

$$(f, v) = \vartheta_{TN}(f, F) \tag{3.2}$$

is equivariant, as well.

Remark 3.2.5. The converse is, in general, false: For example, if (f, v) is a $\rho_t^{(1)}$-equivariant first-order deformation and we define $F = \beta_N(v)$, then (f, F) is a lift of (f, v), but it is not equivariant unless $c(\gamma) \in [\mathfrak{p}]_{\tilde{x}}$ for every $\tilde{x} \in \tilde{M}$.

3.3 First-order harmonic deformations

Definition 3.3.1. Let $f\colon M \to N$ be a smooth map. The *Jacobi operator* $\mathcal{J}\colon \mathcal{C}^\infty(f^*TN) \to \mathcal{C}^\infty(f^*TN)$ is defined as

$$\begin{aligned}\mathcal{J}(v) &= -\mathrm{trace}\Big(\overset{N}{\nabla}\overset{N}{\nabla} v + R^N(\mathrm{d}f(\cdot), v)\mathrm{d}f(\cdot)\Big) \\ &\overset{\mathrm{loc}}{=} -\sum_{i,j} g^{ij}\Big(\frac{D}{\partial x_i}\frac{D}{\partial x_j} v + R^N\big(\frac{\partial f}{\partial x_i}, v\big)\frac{\partial f}{\partial x_j}\Big).\end{aligned}$$

This is the original definition of the Jacobi operator, as in [EL83]. The one we have defined in 1.6.5 coincides with this when we consider the adjoint phls over a symmetric space N, through an application of the isomorphism β_N (recall that in this case the curvature of $\mathrm{d}^{\mathrm{can}}$ corresponds to R^N, and is given by the formula in (1.12)).

Proposition 3.3.2. *Let $\mathcal{F}\colon \tilde{M} \times (-\varepsilon, \varepsilon) \to N$, be any smooth one-parameter family of smooth maps, denote $f_t(\tilde{x}) = \mathcal{F}(\tilde{x}, t)$ and let $v = \frac{\partial f_t}{\partial t}\big|_{t=0}$. Then*

$$\frac{D}{\partial t}\tau_{f_t}\big|_{t=0} = -\mathcal{J}(v),$$

where $\tau_{f_t} \in f_t^*TN \subset \mathcal{F}^*TN$ is the tension field and the covariant differential is given by the pull-back connection of the Levi-Civita connection of N.

Proof. The skeleton of the proof is the following chain of equalities:

$$\begin{aligned}
\frac{D}{\partial t}\sum_s(\overset{N}{\nabla}_{E_s}\mathrm{d}f_t(E_s))\big|_{t=0} &= \sum_s \frac{D}{\partial t}\overset{N}{\nabla}_{E_s}\mathrm{d}f_t(E_s)\big|_{t=0} \\
&\overset{(II)}{=} \sum_s \overset{N}{\nabla}_{E_s}\frac{D}{\partial t}\mathrm{d}f_t(E_s)\big|_{t=0} + R^N(\mathrm{d}f(E_s), v)\mathrm{d}f(E_s) \\
&\overset{(III)}{=} \sum_s \overset{N}{\nabla}_{E_s}\overset{N}{\nabla}_{E_s}v + R^N(\mathrm{d}f(E_s), v)\mathrm{d}f(E_s) = -\mathcal{J}(v).
\end{aligned}$$

Let us explain them in more details. First of all, since $\mathcal{F}$ is defined on a product variety $\tilde{M} \times (-\varepsilon, \varepsilon)$, we may compute the trace (that involves only the "spatial" coordinates $\tilde{x}$) in terms of a fixed orthonormal basis E_s of $T\tilde{M}$. Then (II) follows by definition of curvature (or rather using lemma 3.3.3, part 2., for the curvature of the pull-back connection), since $\frac{\partial}{\partial t}E_s = E_s(\frac{\partial}{\partial t}) = 0$ implies $[X, E_s] = 0$. Finally, (III) is an immediate application of lemma 3.3.3, part 1. (symmetry of covariant derivative and differentiation). □

Lemma 3.3.3. *Let M and N be two Riemannian manifolds, $f\colon M \to N$ a smooth map, and let X, Y be two local vector fields on M such that $[X, Y] = 0$. Denote by ∇ the pull-back of the Levi-Civita connection on N, so that ∇_X and ∇_Y are defined, let R be the curvature tensor on N and $V \in \mathcal{C}^\infty(f^*TN)$ any vector field along f. Then*

1. $\nabla_X \mathrm{d}f(Y) = \nabla_Y \mathrm{d}f(X)$;
2. $\nabla_X\nabla_Y V - \nabla_Y\nabla_X V = R\big(\mathrm{d}f(Y), \mathrm{d}f(X)\big)V$.

Proof. See [dC92], Chap 3, Lemma 3.4 and Chap 4, Lemma 4.1. □

Definition 3.3.4. A first order deformation $v\colon \tilde{M} \to TN$ of a harmonic map $f\colon \tilde{M} \to N$ is said to be harmonic if

$$\mathcal{J}(v) = 0.$$

Example 3.3.5. Proposition 3.3.2 implies that if v comes from a deformation $f_t\colon \tilde{M} \to N$, with f_t harmonic for every $t \in (-\varepsilon, \varepsilon)$, then $v = \frac{\partial f_t}{\partial t}\big|_{t=0}$ is harmonic.

Definition 3.3.6. A map $(f, F)\colon \tilde{M} \to N \times \mathfrak{g}$ is said to be of harmonic type if f is harmonic and

$$J(F) = 0.$$

Remark that, by lemma 1.6.6, this is equivalent to $\Delta(F) = 0$. This is the reason why in that section we worked with arbitrary maps $\tilde{M} \to \mathfrak{g}$ instead of sections of $\mathcal{V}$ (here $\mathcal{V} = \tilde{M} \times_\Gamma \mathfrak{g}$ is the pull-back of the adjoint phls on N).

Lemma 3.3.7. *Let $(f, F)\colon \tilde{M} \to N \times \mathfrak{g}$ be of harmonic type. Then the first-order deformation v of f defined as in* (3.2) *is harmonic, as well.*

Proof. We have already observed that β_N exchanges $\mathcal{J}$ and J. Furthermore, the map defined by

$$\beta_N \circ \vartheta_{TN}\colon N \times \mathfrak{g} \to TN \to N \times \mathfrak{g}$$

is just the projection on $[\mathfrak{p}]$. Hence:

$$\beta_N\big(\mathcal{J}(v)\big) = J\big(\beta_N(v)\big) = J\big(F^{[\mathfrak{p}]}\big) = J(F)^{[\mathfrak{p}]},$$

where for the last equality we have used corollary 1.6.8 applied to the phls on $\tilde{M} \times \mathfrak{g}$ given by $f^*(\mathbb{V}_{ad}, \sigma_{ad}, S_{ad})$. □

3.4 Construction of the harmonic deformations

In this section we want to construct $\rho_t^{(1)}$-equivariant harmonic deformations of a ρ_0-equivariant harmonic map $f\colon \tilde{M} \to N$ by a procedure which in next section will be proved to provide all such deformations. Thanks to lemmas 3.2.4 and 3.3.7, we know that it is enough to construct a map F such that $(f, F)\colon \tilde{M} \to N \times \mathfrak{g}$ is harmonic and $\rho_t^{(1)}$-equivariant.

Lemma 3.4.1. *Let V be a fixed vector space of finite dimension, and $\tau\colon \Gamma = \pi_1(M) \to \mathrm{GL}(V)$ a representation. Denote by $\mathbb{V}$ the associated local system and let $\phi \in Z^1(M, \mathbb{V})$ be a closed 1-form; let $z \in Z^1(\Gamma, V)$ be a 1-cocycle such that the cohomology classes of ϕ and z correspond through the isomorphism $H^1(M, \mathbb{V}) \cong H^1(\Gamma, V)$. Then the set*

$$\big\{F\colon \tilde{M} \to V \ : \ \mathrm{d}F = \tilde{\pi}^*\phi \text{ and } F(\gamma\tilde{x}) = \tau(\gamma)\cdot F(\tilde{x}) + z(\gamma)\big\} \tag{3.3}$$

forms a non-empty torsor over $V^\Gamma = H^0(M, \mathbb{V})$.

Proof. Denote by $\tilde{\phi} = \tilde{\pi}^*\phi$ the pull-back of ϕ to $\tilde{M}$. For any $F\colon \tilde{M} \to V$ such that $\mathrm{d}F = \tilde{\phi}$, define $z_F(\gamma) = z_F(\gamma, \tilde{x}) = F(\gamma\tilde{x}) - \tau(\gamma)\cdot F(\tilde{x})$. This is in fact independent from $\tilde{x}$:

$$\mathrm{d}(z_F(\gamma, \tilde{x})) = \gamma^*\tilde{\phi} - \tau(\gamma)\cdot\tilde{\phi} = 0.$$

Take a base point $\tilde{x}_0 \in \tilde{M}$. The isomorphism of cohomology $H^1(M, \mathbb{V}) \cong H^1(\Gamma, V)$ is then induced by the mapping at the level of cocycles:

$$\begin{aligned} Z^1(M, \mathbb{V}) &\to Z^1(\Gamma, V) \\ \phi &\longmapsto \Big(\gamma \mapsto \int_{\tilde{x}_0}^{\gamma\tilde{x}_0} \tilde{\phi}\Big). \end{aligned}$$

The integral in this expression equals $F(\gamma\tilde{x}_0) - F(\tilde{x}_0)$. Observe that, by definition of z_F,

$$F(\gamma\tilde{x}_0) - F(\tilde{x}_0) = z_F(\gamma) + \tau(\gamma)\cdot F(\tilde{x}_0) - F(\tilde{x}_0) = z_F(\gamma) + \delta\big(F(\tilde{x}_0)\big)(\gamma),$$

where δ denotes the codifferential of group cohomology. Thus z_F differs from z by a coboundary.

Now consider any $\tilde{F}$ such that $\mathrm{d}\tilde{F} = \tilde{\phi}$, so that $F = \tilde{F} + v$, where $v \in V$ is fixed. Then

$$z_F = z_{\tilde{F}} - \delta(v).$$

In particular, every 1-cocycle cohomologous to $z_{\tilde{F}}$ can be obtained in this way, and thus we find a v such that F belongs to the set (3.3). If now F_1 and F_2 are two elements of this set, $F_1 = F_2 + v$ for some fixed $v \in V$, since $\mathrm{d}F_1 = \mathrm{d}F_2$. Further, $\mathrm{d}(v) = 0$, that is, $v \in V^\Gamma$ is fixed by the action of Γ. □

In the following, we shall denote by H the centralizer of the image of ρ_0 and by $\mathfrak{h}$ its Lie algebra. Observe that, under the action of Γ on $\mathfrak{g}$ by $\mathrm{Ad}(\rho_0)$,

$$\mathfrak{h} = H^0(M, \mathrm{Ad}(\rho_0)) = \mathfrak{g}^\Gamma$$

that is, the global sections of the local system $\mathbb{V} = f^*\mathbb{V}_{ad}$.

Now let $c \in Z^1(\Gamma, \mathfrak{g})$ define a first-order deformation of ρ_0, as in definition 3.1.2. By Hodge theory with local coefficients, we can find a harmonic 1-form

$$\omega \in \mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$$

that represents $[c] \in H^1(M, \mathbb{V})$.

Lemma 3.4.2. *Let* $(f, F) \colon \tilde{M} \to N \times \mathfrak{g}$ *be* $\rho_t^{(1)}$*-equivariant and of harmonic type. Then* $\mathrm{d}F = \tilde{\pi}^*\omega$.

Proof. Differentiating the equivariance condition $F(\gamma\tilde{x}) = \mathrm{Ad}_{\rho_0(\gamma)}F(\tilde{x}) + c(\gamma)$ one finds that $\mathrm{d}F$ is the pullback of an $\mathrm{Ad}(\rho_0)$-valued 1-form. By definition, $J(F) = \mathrm{d}^*\mathrm{d}F = 0$, hence $\mathrm{d}F$ is harmonic. To investigate the cohomology class it represents, just integrate and use the equivariance condition once again:

$$\begin{aligned}\int_{\tilde{x}_0}^{\gamma\tilde{x}_0} \mathrm{d}F &= F(\gamma\tilde{x}_0) - F(\tilde{x}_0) = \mathrm{Ad}_{\rho_0(\gamma)}F(\tilde{x}_0) + c(\gamma) - F(\tilde{x}_0) \\ &= \delta\big(F(\tilde{x}_0)\big) + c(\gamma),\end{aligned}$$

where δ denotes group coboundary. Thus $\mathrm{d}F$ represents $[c]$, and by uniqueness $\mathrm{d}F = \tilde{\pi}^*\omega$. □

Proposition 3.4.3. *Let* $\rho_t^{(1)} = (\rho_0, c) \colon \Gamma \to TG$ *be a first order deformation of the representation* ρ_0*. Let* $f \colon \tilde{M} \to N$ *be harmonic and* ρ_0*-equivariant. Take any* F *given by lemma 3.4.1 with* $z = c$ *and* $\phi = \omega$*. Then* $v = \vartheta_{TN}(f, F)$ *is an harmonic and equivariant first order deformation.*

Proof. The F thus constructed is such that (f, F) is $\rho_t^{(1)}$-equivariant and harmonic. This last fact follows from lemma 1.6.6, since

$$0 = \mathrm{d}^*\omega = \mathrm{d}^*\mathrm{d}F = J(F).$$

Then, lemmas 3.2.4 and 3.3.7 give the conclusion. □

3.5 Uniqueness of harmonic deformations

The purpose of this section is to prove that every equivariant and harmonic first order deformation arises by the construction of proposition 3.4.3. In the following, M is equipped with the phls $(\mathcal{V}, \sigma, S)$ which is the pull-back of the adjoint phls on N, as in corollary 1.3.3. Applying corollary 1.6.9 to our setting, we obtain a decomposition $\mathfrak{h} = \mathfrak{h}^{\mathfrak{p}} \oplus \mathfrak{h}^{\mathfrak{k}}$ of the sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$ centralizing the action of Γ.

Proposition 3.5.1. *Every equivariant harmonic first order deformation is constructed by the means of 3.4.3. Furthermore, the set of such deformations is a (non-empty) torsor over* $\mathfrak{h}^{\mathfrak{p}}$.

Proof. Observe that the second statement implies the first one, since we already know how to construct one deformation from a function F as in proposition 3.4.3, that such F's form a torsor over $\mathfrak{h}$ and that passing from F to v "disregards" the $[\mathfrak{k}]$-part. In fact, we only need to prove that the difference of any two harmonic and equivariant first order deformations v, v' is in $\mathfrak{h}^{\mathfrak{p}}$, since the aforementioned proposition grants the existence of one such deformation.

Applying the Maurer-Cartan form β_N to $\mathcal{J}(v) = \mathcal{J}(v') = 0$,

$$\mathrm{d}^*\mathrm{d}\big(\beta_N(v - v')\big) = J\big(\beta_N(v - v')\big) = 0.$$

Since the equivariance condition $v(\gamma\tilde{x}) = c(\gamma)_{f(\gamma\tilde{x})} + \rho(\gamma)_* v(\tilde{x})$ is affine, we have

$$\beta_N(v) - \beta_N(v') = \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v) - \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v'),$$

that is, $\beta_N(v - v')$ is a section of $\mathcal{V}$. An integration by part then gives

$$\mathrm{d}\beta_N(v - v') = 0, \text{ hence } \beta_N(v) = \beta_N(v') + \xi,$$

for some fixed $\xi \in \mathfrak{g}$. Equivariance of v and v' (or rather of their difference) then implies $\xi \in \mathfrak{h}$. On the other hand, ξ has been written as $\beta_N(v) - \beta_N(v') \in [\mathfrak{p}]$, so

$$\xi \in \mathfrak{h} \cap [\mathfrak{p}] = \mathfrak{h}^{\mathfrak{p}}.$$

Conversely, if v is an equivariant and harmonic first-order deformation and $\xi \in \mathfrak{h}^{\mathfrak{p}}$, then $\beta_N(v') = \beta_N(v) + \xi$ gives another equivariant and harmonic first-order deformation, hence the result. ☐

Corollary 3.5.2. *Let* $\rho_t^{(1)} = (\rho_0, c)\colon \Gamma \to TG$ *be a first order deformation of* $\rho_0\colon \Gamma \to G$, *and* $f\colon \tilde{M} \to N$ *a harmonic and* ρ_0*-equivariant mapping. The following conditions are equivalent:*

1. f *admits a unique* $\rho_t^{(1)}$*-equivariant and harmonic deformation to the first order;*
2. $\mathfrak{h}^{\mathfrak{p}} = 0$.

Clearly, these conditions are implied by $\mathfrak{h} = 0$, *and, in turn, by asking* $\mathrm{Image}(\rho_0)$ *to be Zariski-dense.*

3.6 Conclusions

Summing up the results of this section, we have proved:

Theorem 3.6.1. *Let M be a compact Riemannian manifold, $\mathbb{G}$ an algebraic reductive group, $G = \mathbb{G}(\mathbb{R})$ the Lie group of its real points, $\rho_t^{(1)} = (\rho_0, c)\colon \pi_1(M) \to G$ a first-order deformation of ρ_0 and $f\colon \tilde{M} \to N$ a harmonic and ρ_0-equivariant map. Then every first order harmonic and $\rho_t^{(1)}$-equivariant deformation v of f comes from a harmonic 1-form $\omega \in \mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$ representing $[c]$. More precisely, the map*

$$\left\{ \begin{array}{c} F\colon \tilde{M} \to \mathfrak{g} \;:\; \mathrm{d}F = \omega \text{ is harmonic} \\ \text{and } F(\gamma\tilde{x}) = \mathrm{Ad}_{\rho_0(\gamma)} F(\tilde{x}) + c(\gamma) \end{array} \right\} \xrightarrow{\vartheta_{TN}} \left\{ \begin{array}{c} v \in \mathcal{C}^\infty(f^*TN) \text{ harmonic} \\ \text{and } \rho_t^{(1)}\text{-equivariant.} \end{array} \right\}$$

is affine and surjective, and corresponds to the linear projection on the associated vector spaces:

$$\mathfrak{h} = H^0(M, \mathrm{Ad}(\rho_0)) \longrightarrow H^0(M, \mathrm{Ad}(\rho_0)) \cap \mathfrak{p} = \mathfrak{h}^{\mathfrak{p}}.$$

Thanks to this theorem, in the following sections we will be able to translate every appearance of the variation of f (e.g. to compute the variation of the energy) in terms of the harmonic 1-form ω. To conclude, we will prove the following functoriality statement with respect to holomorphic maps between Kähler manifolds:

Proposition 3.6.2. *Let X, X' be Kähler manifolds, $\varphi\colon X' \to X$ a holomorphic map. Let $\rho_t^{(1)} = (\rho_0, c)\colon \pi_1(X) \to G$ be a first order deformation of ρ_0, and define $\rho_t^{(1)'} = (\rho_0', c') = \rho_t^{(1)} \circ \varphi_*$. Let $f\colon \tilde{X} \to N$ be a harmonic and ρ_0-equivariant map, and define $f' = f \circ \varphi$. Then f' is harmonic and ρ_0'-equivariant, and the construction of theorem 3.6.1 is functorial with respect to φ, that is, we have a commutative diagram:*

$$\begin{array}{ccc} \left\{ \begin{array}{c} F\colon \tilde{X} \to \mathfrak{g} \;:\; \mathrm{d}F = \omega \text{ is harmonic} \\ \text{and } F(\gamma\tilde{x}) = \mathrm{Ad}_{\rho(\gamma)} F(\tilde{x}) + c(\gamma) \end{array} \right\} & \xrightarrow{\vartheta_{TN}} & \left\{ \begin{array}{c} v \in \mathcal{C}^\infty(f^*TN) \text{ harmonic} \\ \text{and } \rho_t^{(1)}\text{-equivariant.} \end{array} \right\} \\ \downarrow \varphi^* & & \downarrow \varphi^* \\ \left\{ \begin{array}{c} F'\colon \tilde{X}' \to \mathfrak{g} \;:\; \mathrm{d}F' = \omega' \text{ is harmonic} \\ \text{and } F'(\gamma\tilde{x}) = \mathrm{Ad}_{\rho(\gamma)} F'(\tilde{x}) + c'(\gamma) \end{array} \right\} & \xrightarrow{\vartheta_{TN}} & \left\{ \begin{array}{c} v' \in \mathcal{C}^\infty(f'^*TN) \text{ harmonic} \\ \text{and } \rho_t^{(1)'}\text{-equivariant.} \end{array} \right\} \end{array}$$

Remark that ρ_0'-equivariance of f' is obvious, and harmonicity is the content of lemma 1.8.1.

Proof. First notice that the vertical arrow on the left is well-defined thanks to corollary 1.8.6: If $F\colon \tilde{X} \to \mathfrak{g}$ is such that $\mathrm{d}F = \omega$, letting $F' = F \circ \varphi$ we have $J(F') = \mathrm{d}^*\varphi^*\omega = 0$. Equivariance follows from the definition of $\rho_t^{(1)'}$: For every $\gamma \in \pi_1(X')$,

$$\begin{aligned} F'(\gamma\tilde{x}) &= F\big(\varphi(\gamma\tilde{x})\big) = F\big(\varphi_*(\gamma)\cdot\varphi(\tilde{x})\big) = \mathrm{Ad}_{\rho_0(\varphi_*\gamma)}F\big(\varphi(\tilde{x})\big) + c\big(\varphi_*\gamma\big) \\ &= \mathrm{Ad}_{\rho_0'(\gamma)}F'(\tilde{x}) + c'(\gamma). \end{aligned} \tag{3.4}$$

Theorem 3.6.1 implies that the horizontal arrows are surjective. The good definition of the φ^* on the right follows by combining this same theorem with corollary 1.8.6: If $v \in \mathcal{C}^\infty(f^*TN)$ is a harmonic and $\rho_t^{(1)}$-equivariant first-order deformation of f, and $v' = \varphi^*v$, equivariance of v' follows as in (3.4). For harmonicity, we prove that $J(\beta_N(v')) = 0$ follows from $\varphi^*(J(\beta_N(v)) = 0$: Let $F\colon \tilde{X} \to \mathfrak{g}$ be such that $J(F) = 0$ and $F^{[\mathfrak{p}]} = \beta_N(v)$, and define $F' = \varphi^*(F)$, so that $F'^{[\mathfrak{p}]} = \beta_N(v')$. Then:

$$J(\beta_N(v')) = J\big(F'^{[\mathfrak{p}]}\big) = J\big(\varphi^*(F)^{[\mathfrak{p}]}\big) = J\big(\varphi^*(F)\big)^{[\mathfrak{p}]} = \varphi^*\big(J(F)\big)^{[\mathfrak{p}]},$$

thanks to the commutativity of J and projection on $[\mathfrak{p}]$ (corollary 1.6.8) and that of J and φ^*, provided that $\mathrm{d}F$ is an $\mathrm{Ad}(\rho_0)$-valued 1-form.

The commutativity of the diagram in proposition 3.6.2 is clear. □

3.7 Example: the case $G = GL(1,\mathbb{C})$

To give a description of the result in the easiest possible case, in this section we briefly analyze the special case $G = GL(1,\mathbb{C})$. Explicitely, $G = GL(1,\mathbb{C}) = \mathbb{C}^*$, $K = U(1)$ and $N = \mathbb{R}_{>0}$. The main simplification here is that the group (and the Lie algebra) are abelian. If $f\colon \tilde{M} \to N$ is a mapping, let $g\colon \tilde{X} \to \mathbb{R}$ be given by $g = \log(f)$.

Once a representation and a harmonic and equivariant f are given, the structure of section 1 becomes much easier, since inside $\mathfrak{g} = \mathbb{C}$ we have $[\mathfrak{p}] = \mathfrak{p} = \mathbb{R}$ and $[\mathfrak{k}] = \mathfrak{k} = i\mathbb{R}$. The scalar product on $\tilde{M} \times \mathfrak{g}$ is constant, that is, independent of the function f and the point $\tilde{x} \in \tilde{M}$; since the Lie algebra is abelian, $\mathrm{d}^{\mathrm{can}} = \mathrm{d}$ gives the metric connection on $\mathcal{V} = M \times \mathfrak{g}$. The 1-form $\tilde{\beta}$ is simply $\mathrm{d}g$, as it may be seen for example by means of lemma 1.5.9, since $\mathrm{d}f\cdot f^{-1} = \frac{\exp g\cdot \mathrm{d}g}{\exp g} = \mathrm{d}g$ (here we do not need to take a section of f, and consequently no projection on $\mathfrak{p}$, either).

Now fix a representation $\rho_0\colon \Gamma \to \mathbb{C}^*$. Write $\rho_0 = \rho_c \cdot \rho_{nc}$ where $\rho_c(\gamma) \in U(1)$ and $\rho_{nc}(\gamma) = |\rho_0(\gamma)| \in \mathbb{R}_{>0}$. Then f is ρ_0-equivariant when $f(\gamma\tilde{x}) = \rho_{nc}(\gamma)f(\tilde{x})$, and g when $g(\gamma\tilde{x}) = \log\rho_{nc}(\gamma) + g(\tilde{x})$. Define $r\colon \Gamma \to \mathbb{R}$ as $r = \log(\rho_{nc})$, so that r gives a 1-cohomology class on M. Since the logarithm is an isometry between N and $\mathbb{R}$, the mapping f is harmonic if and only if g is a harmonic function; in particular, every two harmonic and equivariant g's differ by a constant (the difference is harmonic and defined on M, which is compact). To construct such a g, take any harmonic 1-form β representing the cohomology class given by r, let $\tilde{\beta}$ be its pull-back to $\tilde{M}$, and define

$$g(\tilde{x}) = \int_{\tilde{x}_0}^{\tilde{x}} \beta.$$

Then $\mathrm{d}g = \tilde{\beta}$, as wanted, which is harmonic by hypothesis, so that g is a harmonic function.

In this case, the objects appearing in theorem 3.6.1 are as follows: We have two functions $F\colon \tilde{M} \to \mathbb{C}$ and $\beta_N(v)\colon \tilde{M} \to \mathbb{R}$. The definitions of harmonic type for F and harmonic for v reduce simply to both being harmonic functions. Equivariance conditions for F and v are the same, the latter involving $c(\gamma)^{[\mathfrak{p}]} = \mathcal{R}e(c(\gamma))$. Clearly, the sets consisting of the F's and the one consisting of the v's are in correspondence through ϑ_{TN}, which is essentially $\mathcal{R}e$. Finally, $\mathrm{Ad}(\rho_0)$ is trivial, hence $\mathfrak{h} = H^0(M, \mathbb{C}) = \mathbb{C}$, and $\mathfrak{h}^{\mathfrak{p}} = \mathbb{R}$. The difference of any two $F's$ or $v's$ being defined on M, the last statement of the theorem only expresses that such differences must be constant.

When $M = X$ is a Kähler manifold, one can further consider the holomorphic 1-form α representing the cohomology class, thanks to the Dolbeault isomorphism. Then one simply has $\beta = \mathcal{R}e(\alpha)$ and $\theta = \frac{1}{2}\alpha$. Harmonicity of these forms are a consequence of the (classical) Kähler identities.

Chapter 4

First variation of the energy: $\mathbb{C}$-VHS as critical points

Introduction au chapitre

Ce chapitre se concentre sur l'étude des variations du premier ordre de la fonctionnelle de l'énergie sur l'espace de modules des représentations semi-simples. Plus en général, on se donne une déformation au premier ordre v, harmonique et $\rho_t^{(1)}$-équivariante, d'une métrique harmonique f et on étudie la variation de l'énergie le long de v. Pour le faire, on définit l'énergie de (f, v) comme le premier ordre de l'énergie d'une famille f_t telle que $\frac{\partial f_t}{\partial t}\big|_{t=0} = v$. Ensuite, une application directe du théorème du chapitre 3 permet d'identifier ce premier ordre avec le produit scalaire L^2 entre ω, la 1-forme harmonique induite par $\rho_t^{(1)}$, et β (la 1-forme correspondante à $\mathrm{d}f$). En fait, comme $\mathrm{d}^*\beta = 0$, le résultat ne changerait pas en remplaçant ω par n'importe quelle autre 1-forme harmonique représentant $\rho_t^{(1)}$.

Des cas particuliers sont examinés: si $M = X$ est kählerienne, ce premier ordre a une interpretation cohomologique ressemblante à celle usuelle pour l'énergie; dans le cas abelien, l'étude des points critiques de l'énergie est très simple, car les seuls points critiques sont des minima globaux. Le reste du chapitre est dédié à la preuve du théorème principal, qui, sous l'hypothèse de Kähler sur $M = X$, identifie les points critiques de l'énergie avec les variations complexes de structures de Hodge. Ceci est une généralisation du même résultat, déjà connu dans le cas où X est une surface de Riemann et l'espace de modules est lisse (cfr. [Hit87]). D'un côté, les variations de structures de Hodge complexes sont des points critiques de l'énergie en conséquence du fait que le générateur infinitesimal γ de l'action de S^1 a pour espaces propres les facteurs de la décomposition de Hodge induite sur les endomorphismes, ce qui permet d'identifier β avec $D^c\gamma$. On obtient l'autre identification en considerant la variation infinitesimale induite par l'action de $\mathbb{C}^*$. On remarque que, comme la variation de l'énergie peut être calculée en remplaçant ω par n'importe quelle autre 1-forme dans sa classe de cohomologie, le théorème est valable aussi dans les points singuliers, en définissant un point critique celui qui annulle les dérivées le long de toutes directions dans le tangent de Zariski de $\mathrm{Hom}(\Gamma, G)$.

4.1 First variation of the energy

Here M denotes a compact Riemannian manifold. We want to make use of theorem 3.6.1 to give a formula for the first variation of the energy.

Lemma 4.1.1. *Let $\rho_t^{(1)} = (\rho_0, c)\colon \Gamma \to TG$ be a first order deformation of ρ_0 and $v \in \mathcal{C}^\infty(f^*TN)$ a $\rho_t^{(1)}$-equivariant first order deformation of f. Then the quantity*

$$\big\langle \overset{N}{\nabla} v, \mathrm{d}f \big\rangle$$

defines a Γ-invariant function on $\tilde{M}$.

Proof. We can equivalently prove the result after applying β_N to both sides, that is, we want to prove that

$$\big\langle \overset{\mathrm{can}}{\nabla} \beta_N(v), \tilde{\beta} \big\rangle(\gamma\tilde{x}) = \big\langle \overset{\mathrm{can}}{\nabla} \beta_N(v), \tilde{\beta} \big\rangle(\tilde{x}). \tag{4.1}$$

Applying β_N (which is a right inverse of ϑ_{TN}) to definition 3.2.1 we get $\beta_N(v)_{\gamma\tilde{x}} = \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v)_{\tilde{x}} + c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}$. Hence the term on the left hand side of (4.1) equals

$$\Big\langle \overset{\mathrm{can}}{\nabla} \big(\mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v) + c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\big), \mathrm{Ad}_{\rho_0(\gamma)}\tilde{\beta} \Big\rangle_{\gamma\tilde{x}}.$$

Since, by definition of the metric (1.4) and of the canonical connection:

$$\Big\langle \overset{\mathrm{can}}{\nabla} \big(\mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v)\big), \mathrm{Ad}_{\rho_0(\gamma)}\tilde{\beta} \Big\rangle_{\gamma\tilde{x}} = \big\langle \overset{\mathrm{can}}{\nabla} \beta_N(v), \tilde{\beta} \big\rangle_{\tilde{x}},$$

we only need to prove the vanishing of $\langle c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}, \mathrm{Ad}_{\rho_0(\gamma)}\tilde{\beta}\rangle$. As $\overset{\mathrm{can}}{\nabla}$ commutes with projection on $[\mathfrak{p}]$ and $c(\gamma)$ is constant, that is, $\mathrm{d}\big(c(\gamma)\big) = \mathrm{d}^{\mathrm{can}}c(\gamma) + [\tilde{\beta}, c(\gamma)] = 0$, we have

$$\overset{\mathrm{can}}{\nabla} c(\gamma)^{[\mathfrak{p}]} = -\big[\tilde{\beta}, c(\gamma)\big]^{[\mathfrak{p}]} = -\big[\tilde{\beta}, c(\gamma)^{[\mathfrak{k}]}\big].$$

Thus:

$$\Big\langle \Big(- \big[\tilde\beta, c(\gamma)\big]\Big)^{[\mathfrak{p}]}_{\gamma\tilde x}, \tilde\beta_{\gamma\tilde x}\Big\rangle_{\gamma\tilde x} = -\Big\langle \big[\tilde\beta_{\gamma\tilde x}, c(\gamma)\big], \tilde\beta_{\gamma\tilde x}\Big\rangle_{\gamma\tilde x}$$
$$= -\sum_{i,j} g^{ij}\Big\langle c(\gamma), \Big[\tilde\beta_{\gamma\tilde x}\big(\frac{\partial}{\partial x_i}\big), \tilde\beta_{\gamma\tilde x}\big(\frac{\partial}{\partial x_j}\big)\Big]\Big\rangle_{\gamma\tilde x} = 0.$$

□

Definition 4.1.2. Let $\rho_t^{(1)}$ be a first order deformation of a representation ρ_0, and v a $\rho_t^{(1)}$-equivariant first order deformation of a ρ_0-equivariant map $f\colon \tilde M \to N$. We define its energy as

$$E_t = E(f, v) = E(f) + t\int_M \big\langle \overset{N}{\nabla} v, \mathrm{d}f\big\rangle \in \mathbb{R}[t]/(t^2). \tag{4.2}$$

When f_t is defined for $t \in (-\varepsilon, \varepsilon)$, this definition agrees with the first order jet of $t \mapsto E(f_t)$, since

$$\frac{\partial}{\partial t}E(f_t)\Big|_{t=0} = \frac{1}{2}\int \frac{\partial}{\partial t}\|\mathrm{d}f_t\|^2 = \int \big\langle \frac{D}{\partial t}\mathrm{d}f, \mathrm{d}f\big\rangle = \int \big\langle \nabla v, \mathrm{d}f\big\rangle,$$

where for the last equality we have used lemma 3.3.3. Composing with $\beta_N\colon TN \to [\mathfrak{p}] \subseteq N \times \mathfrak{g}$ we obtain:

$$E(f_t) = \frac{1}{2}\int \|\tilde\beta\|^2 + t\int \big\langle \overset{\mathrm{can}}{\nabla}\beta_N(v), \tilde\beta\big\rangle. \tag{4.3}$$

Proposition 4.1.3. *Let $\rho_t^{(1)} = (\rho_0, c)\colon \pi_1(M) \to TG$ a first order deformation of ρ_0, and $f_t\colon \tilde M \to N$ a $\rho_t^{(1)}$-equivariant, harmonic first-order deformation of f. Then*

$$\frac{\partial E_t}{\partial t}\Big|_{t=0} = \int_M \big\langle \omega, \beta\big\rangle \mathrm{dVol}_g, \tag{4.4}$$

where ω is the harmonic representative of $[c]$.

Proof. Thanks to theorem 3.6.1, there is an F such that (f, F) is $\rho_t^{(1)}$-equivariant, $\mathrm{d}F = \omega$ and $\vartheta_{TN}(f, F) = v$. Fix one such F, so that in particular $\beta_N(v) = F^{[\mathfrak{p}]}$. Then, pulling back to $\tilde M$, the right hand side of (4.4) can be written as

$$\int_{\tilde M/\Gamma} \langle\tilde\omega, \tilde\beta\rangle = \int \langle\tilde\omega^{[\mathfrak{p}]}, \tilde\beta\rangle = \int \big\langle \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]}, \tilde\beta\big\rangle + \big\langle [\tilde\beta, F^{[\mathfrak{k}]}], \tilde\beta\big\rangle.$$

The second summand vanishes because the Lie bracket by $F^{[\mathfrak{k}]}$ is anti self-adjoint. Since $\beta_N(v) = F^{[\mathfrak{p}]}$, this concludes the proof, using (4.3). □

We list some examples to illustrate how the energy functional behaves (in particular, describing its critical points) in several special cases.

Example 4.1.4. When $M = X$ is a Kähler manifold, equation (4.4) is independent of the particular metric chosen on $X = M$ in its Kähler class. This follows just as we did in equation (1.16) for the energy of f, since in this case (4.4) can be rewritten as

$$\frac{\partial E_t}{\partial t}\Big|_{t=0} = \int_X \big\langle \omega \wedge *\beta \big\rangle = -\frac{1}{n!}\int_X \operatorname{trace}\big(\omega \wedge (\theta^* - \theta)\big) \wedge \Omega^{n-1},$$

where, for the last equality, we have used that $*\beta = (\theta^* - \theta) \wedge \Omega^{n-1}$ takes values in the anti self-adjoint part of $\mathfrak{g} \otimes \mathbb{C}$, hence the scalar product equals the Killing form, up to a negative constant. We need to prove that $\operatorname{trace}(\omega \wedge (\theta - \theta^*))$ is closed. Since $\mathrm{d}\omega = 0$, and recalling that $\mathrm{d}\theta = \mathrm{d}(\theta^*) = [\theta, \theta^*]$, the conclusion is immediate. This same result will also follow from the more general lemma 6.1.6.

Example 4.1.5. We now continue the discussion of the case $G = \mathbb{C}^*$ from section 3.7, keeping the same notations therein. To make the discussion simplier and independent from previous results, we will assume everything to be defined for $t \in (-\varepsilon, \varepsilon)$.

Consider $\rho_t \colon \Gamma \to \mathbb{C}^*$, a deformation of ρ_0, suppose $f_t \colon \tilde{M} \to \mathbb{R}_{>0}$ to be harmonic and ρ_t-equivariant, and define $\tilde{\beta}_t$ consequently (we have seen in section 3.7 that $\tilde{\beta}_t = \mathrm{d}\log(f_t)$). Then the real part of the harmonic 1-form ω is simply $\omega^{[\mathfrak{p}]} = \frac{\partial \tilde{\beta}_t}{\partial t}\big|_{t=0}$. To see this, first notice that in this case $\omega^{[\mathfrak{p}]}$ is the real part of ω, hence it is the harmonic representative of the real part of the 1-cocycle c. Harmonicity of $\frac{\partial \tilde{\beta}_t}{\partial t}\big|_{t=0}$ follows from that of $\tilde{\beta}_t$, for every t (since derivations commute): On the one hand, $\mathrm{d}^*\tilde{\beta}_t = 0$ is one way to express harmonicity of f_t and on the other hand $\tilde{\beta}_t = \mathrm{d}\log(f_t)$ implies $\mathrm{d}\tilde{\beta}_t = 0$. Now equation (4.4) follows from the flat derivation being metric:

$$\frac{\partial}{\partial t}\Big|_{t=0} \frac{1}{2}\int \|\beta_t\|^2 = \int \Big\langle \frac{\partial}{\partial t}\tilde{\beta}_t\Big|_{t=0}, \tilde{\beta} \Big\rangle = \int \langle \omega^{[\mathfrak{p}]}, \beta \rangle = \int \langle \omega, \beta \rangle.$$

Finally, when $M = X$ is assumed to be Kähler, the Betti and Dolbeault moduli spaces (cfr. [Sim94] for the terminology) are very easy to describe, as is the homeomorphism between them and the $\mathbb{C}^*$-action:

$$\mathbb{M}_{\mathrm{B}}(X, \mathbb{C}^*) = \operatorname{Hom}(\Gamma, \mathbb{C}^*) \cong \operatorname{Pic}^0(X) \times H^0(\Omega^1) = \mathbb{M}_{\mathrm{Dol}}(X, \mathbb{C}^*),$$

where the moduli space of Higgs bundles splits as the product of $\operatorname{Pic}^0(X)$, the moduli space of degree 0 holomorphic line bundles, and the space of global holomorphic 1-forms $H^0(\Omega^1)$. Then the fixed points of $(\mathcal{E},\theta) \mapsto (\mathcal{E}, t\theta)$ are only those with $\theta = 0$, so these are the $\mathbb{C}$-VHS. In particular, every $\mathbb{C}$-VHS has $\beta = \theta + \theta^* = 0$, so that the energy vanishes (i.e. f is constant) and they are clearly critical points (indeed, the only ones, and they are also minima) of the energy.

Example 4.1.6. When $M = \Sigma$ is a Riemann surface of genus $g > 2$, the energy functional has been intensively studied in [Hit87] and the subsequent literature (cfr. e.g. [Hit92]). Up to some constant, there a functional on the moduli space of Higgs bundles is defined by

$$(\mathcal{E},\theta) \mapsto \int_\Sigma \|\theta\|^2.$$

This coincides with our definition $\frac{1}{2}\int_M \|\beta\|^2$ since $\beta = \theta + \theta^*$, and also θ and θ^* are adjoint, hence have the same norm.

In the case of a Riemann surface, the energy functional gives a proper functional on the moduli space of Higgs bundles (hence, on that of representations). Furthemore, it is a moment map for the circle action $e^{it} \cdot (\mathcal{E},\theta) = (\mathcal{E}, e^{it}\theta)$. In particular, the smooth critical points are fixed points of this action, that is, $\mathbb{C}$-VHS. We will generalize this last result in the upcoming sections. Results about the second order derivative of f (index, or positivity) will be analyzed in section 6.

Example 4.1.7. To conclude our list of examples, let us resume the example of $M = S^1$ so that $\Gamma = \mathbb{Z}$ and a representation is just the data of an element $g = \rho_0(1)$. A deformation of the representation g is, in this case, just an element $\xi \in \mathfrak{g}$ of the Lie algebra of G; this gives the group 1-cocycle $c\colon \mathbb{Z} \to \mathfrak{g}$ by defining, for $n > 0$,

$$c(n) = \xi + \mathrm{Ad}_g\xi + \cdots + \mathrm{Ad}_{g^{n-1}}\xi,$$

and similarly up to $n+1$ if $n < 0$. The 1-coboundaries are given by

$$\delta(\eta)(n) = \mathrm{Ad}_{g^n}\xi - \xi.$$

Hence the 1-cohomology of $\mathbb{Z}$ with values in $\mathfrak{g}$ identifies with $\mathfrak{g}/(\mathrm{Ad}_g - 1)\mathfrak{g}$. An equivariant and harmonic first order deformation v is a vector field along f that satisfies:

$$J(\beta_N(v)) = 0; \quad v(x+1) = g_* v(x) + \vartheta_{TN}(f(x), \xi).$$

By the first equation it is a Jacobi field (hence the name of J). In this case, in terms of a global coordinate x on $\mathbb{R}$, ω may be written as $\omega = \eta(x)\mathrm{d}x$, and the harmonicity condition becomes

$$\mathrm{d}^*\omega = -\frac{\partial\eta(x)}{\partial x} + 2\Big[\beta\big(\frac{\partial f(x)}{\partial x}\big), \eta(x)\Big] = 0.$$

Then, the first order derivative of the energy functional (that is, of the square of the translation length) may be expressed in terms of the Killing form as

$$\frac{\partial\mathcal{E}(g_t)}{\partial t}\Big|_{t=0} = \int_0^1 \mathrm{Kill}\Big(\eta(x), \beta\big(\frac{\partial f(x)}{\partial x}\big)\Big)\mathrm{d}x.$$

4.2 $\mathbb{C}$-VHS are critical points of the energy functional

Let X be a compact Kähler manifold, $G = K^{\mathbb{C}}$ a complex semisimple algebraic group (here K is a maximal compact subgroup), and $\rho_t^{(1)} = (\rho_0, c)\colon \Gamma \to TG$, where $t^2 = 0$, a first order deformation of ρ_0, which is supposed to be induced by a $\mathbb{C}$-VHS. Suppose further that (f, v) is a first order $\rho_t^{(1)}$-equivariant and harmonic deformation of $f = f_0$, which is induced by the period mapping $\Phi\colon \tilde{M} \to G_0/V_0$ (notations are coherent with those of 1.2.5). The purpose of this section is to prove that then f_0 is a critical point for the energy $E(f_t) = E(f) + t\int\langle\overset{N}{\nabla}v, \mathrm{d}f\rangle$, i.e. that $\frac{\partial E_t}{\partial t}\big|_{t=0} = 0$. First of all, we remark that in the case when X is a smooth projective variety, we can deduce this result from the case of curves (cfr. example 4.1.6):

Proposition 4.2.1. *Let X be a smooth projective variety, $G = K^{\mathbb{C}}$ a complex algebraic group. Then $\rho_0\colon \Gamma \to G$ is a critical point for the energy if and only if it is a fixed point of the S^1-action on the moduli space.*

Proof. When $\dim_{\mathbb{C}} X = 1$, this is due to Hitchin (cfr. example 4.1.6). We reduce the general case to this one. Let $\dim_{\mathbb{C}} X = n$, and Ω be a Kähler form. By projectivity, Ω may be supposed integral and very ample, so that we find $n-1$ general hyperplane sections $H_1, \ldots, H_{n-1}$ in the class of Ω. Write C for the complete intersection of $H_1, \ldots, H_{n-1}$, so that C is a smooth curve such that $[\Omega] = [C]$ in $H^2(X, \mathbb{Z})$. In particular, the formula for the energy (1.16) may be expressed in terms of a Higgs bundle $(\mathcal{E}, \theta)$ corresponding to ρ_0 as:

$$E(\rho_0) = \int_X \mathrm{trace}\big(\theta \wedge \theta^*\big) \wedge \Omega^{n-1} = \int_C \mathrm{trace}\big(\theta\big|_C \wedge \theta\big|_C^*\big).$$

Now Simpson ([Sim92], sections 1 and 4) proves that the construction associating a Higgs bundle to a representation is functorial with respect to pull-backs, and also that, when $i\colon C \hookrightarrow X$ is a complete intersection, ρ_0 is induced by a $\mathbb{C}$-VHS if and only if $i^*\rho_0$ is. These two facts together complete the proof. □

For the general Kähler case, we need some preparation.

Lemma 4.2.2. *Let $f_0\colon \tilde{X} \to G_0/K_0$ be induced by the period mapping Φ associated to a $\mathbb{C}$-VHS $(\mathbb{V}, D, S)$, fix a base point $x_0 \in X$ and let G be a complex subgroup of $\mathrm{GL}(\mathcal{V}_{x_0})$ containing G_0. Then, denoting by $\mathfrak{g}$ the Lie algebra of G, the flat bundle $\tilde{X} \times \mathfrak{g}$ has a Hodge decomposition of weight zero, compatible with the Lie bracket, which we write as*

$$\tilde{X} \times \mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \left[\mathfrak{g}^{p,-p}\right].$$

Proof. We will regard G_0 as a group of automorphisms of $\mathcal{V}_{x_0}$. Then $\mathfrak{g}_0$, its Lie algebra, and its complexification $\mathfrak{g}_{0,\mathbb{C}}$ both consist of endomorphisms of $\mathcal{V}_{x_0} = \bigoplus V^{r,s}$. We can decompose the complexified Lie algebra according to how it behaves on the Hodge structure:

$$\mathfrak{g}_{0,\mathbb{C}}^{p,-p} = \left\{\xi \in \mathfrak{g}_{0,\mathbb{C}} \ :\ \xi \cdot V^{r,s} \subseteq V^{r+p,s-p}\right\}.$$

This gives a $\mathbb{C}$-Hodge structure of weight 0 on $\mathfrak{g}_{0,\mathbb{C}}$. We define an action $S^1 \to \mathrm{SL}(\mathfrak{g}_{0,\mathbb{C}})$ by

$$z \cdot \xi^{p,-p} = z^{2p}\xi^{p,-p}. \tag{4.5}$$

Thanks to this action, we extend the Hodge structure to all of $\mathfrak{g}$ via the embedding $\mathrm{SL}(\mathfrak{g}_{0,\mathbb{C}}) \subseteq \mathrm{SL}(\mathfrak{g})$:

$$\mathfrak{g}^{p,-p} = \left\{\xi \in \mathfrak{g} \ :\ z \cdot \xi = z^{2p}\xi^{p,-p}\right\}.$$

Now we define the $\mathbb{C}$-VHS structure on $f^*(N \times \mathfrak{g}) = \tilde{X} \times \mathfrak{g}$ by letting

$$\tilde{X} \times \mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \left[\mathfrak{g}^{p,-p}\right],$$

where $\left[\mathfrak{g}^{p,-p}\right] = \mathrm{Ad}_{\Phi(\tilde{x})}(\mathfrak{g}^{p,-p})$. Notice that here we need to take adjunction by Φ instead of f, because the Hodge decomposition on $N \times \mathfrak{g}$ is only V_0-invariant, not K_0-invariant. □

Remark 4.2.3. Note that, in general, this decomposition will not be compatible with the pull-back of the adjoint phls to give a $\mathbb{C}$-VHS structure on $\tilde{X} \times \mathfrak{g}$, unless G is the complexification of G_0.

Lemma 4.2.4. *With the same notations as in the previous lemma, let $\mathfrak{v}_0$ be the Lie algebra of V_0. Then there exists a parallel section γ of $[\mathfrak{v}_0]$ such that, for every $\xi \in \mathcal{C}^\infty(\tilde{X} \times \mathfrak{g})$, we have*

$$[\gamma, \xi] = \sum_p ip\xi^{-p,p}, \quad \textit{where } \xi = \sum_p \xi^{-p,p},\ \xi^{-p,p} \in [\mathfrak{g}^{-p,p}].$$

Proof. Let $(\mathcal{E}, \theta)$ be the Higgs bundle corresponding to ρ_0 via the harmonic metric f_0, so that there is a decomposition $\mathcal{E} = \bigoplus \mathcal{E}^{r,s}$. Consider the action of $\mathbb{C}^*$ on $\mathcal{E}$ given by multiplication by t^r on $\mathcal{E}^{r,s}$. Then, Simpson proves in [Sim92], Lemma 4.4, that the restriction of this action to $U(1)$ induces a 1-parameter family of automorphisms of G_0. Denote by $\phi_\theta \in \mathrm{Aut}(G_0)$ the action given by $e^{-i\theta}$; then, since $\phi_0 = \mathrm{id}$ and since G_0 being reductive implies that the map $G^\circ \to \mathrm{Aut}(G_0)^\circ$ is surjective (cfr. Simpson, loc. cit., 4.4.1. and 4.4.2.), we have $\phi_\theta = \mathrm{Ad}_{g_\theta}$ for some g_θ in the identity component of G_0. Thanks to the $\mathrm{Ad}(V_0)$-invariance of the action of g_θ we can define, for every $\tilde{x} \in \tilde{X}$,

$$\gamma(\tilde{x}) = \mathrm{Ad}_{\Phi(\tilde{x})}\big(\gamma_{x_0}\big), \quad \text{where } \gamma_{x_0} = \frac{\partial g_\theta}{\partial \theta}\Big|_{\theta=0} \in \mathfrak{g}_0.$$

This is in fact a section of $[\mathfrak{v}_0] \to X$ and parallel with respect to $\mathrm{d}^{\mathrm{can}}$, simply because we constructed it by parallel transport (here we use the first definition of $\mathrm{d}^{\mathrm{can}}$ in definition 1.5.6). From the definition of the Hodge decomposition on $\tilde{X} \times \mathfrak{g}$, it follows that $\mathrm{Ad}_{g_\theta}(\xi^{-p,p}) = e^{ip\theta}\xi^{p,-p}$ for every $\xi^{p,-p} \in [\mathfrak{g}^{-p,p}]$. Thus, $\mathrm{ad}_\gamma(\xi^{-p,p}) = ip\xi^{-p,p}$; in particular, $\gamma \in [\mathfrak{g}^{0,0}]$, and since $\gamma \in [\mathfrak{g}_0]$, we have also $\gamma \in [\mathfrak{v}_0]$. □

Lemma 4.2.5. *Let ρ_0 be induced by a $\mathbb{C}$-VHS and $f_0\colon \tilde{X} \to G/K$ be induced by the period map $\Phi\colon \tilde{X} \to G_0/V_0$, where $G_0 \subseteq G \subseteq \mathrm{GL}(\mathcal{V}_{x_0})$, as above. Then*

$$\begin{aligned} \theta &= iD''\gamma; \\ \theta^* &= -iD'\gamma. \end{aligned} \tag{4.6}$$

Proof. Since $\mathrm{d}^{\mathrm{can}}\gamma = 0$, both $\partial\gamma$ and $\bar{\partial}\gamma$ vanish. Hence

$$D'\gamma = \partial\gamma + [\theta^*, \gamma] = -[\gamma, \theta^*]; \quad D''\gamma = \bar{\partial}\gamma + [\theta, \gamma] = -[\gamma, \theta].$$

Now $[\gamma, \theta] = i\theta$, since $\theta \in \mathcal{A}^1([\mathfrak{g}^{-1,1}])$, and $[\gamma, \theta^*] = -i\theta^*$ so we obtain (4.6). □

Corollary 4.2.6. *When f_0 comes from a VHS, we may write*

$$\beta = D^c\gamma,$$

where $D^c = -i(D'' - D')$ is the real operator constructed with respect to the total Weil operator C (cfr. [Zuc79]).

Corollary 4.2.7. *Let X be a compact Kähler manifold, $\rho_t^{(1)}$ a first order deformation of a representation $\rho_0 \colon \Gamma = \pi_1(X) \to G$, and v a first order harmonic and $\rho_t^{(1)}$-equivariant deformation of $f_0 \colon \tilde{X} \to G/K$. Suppose that ρ_0 and $f_0 = \pi \circ \Phi$ are induced by a $\mathbb{C}$-VHS as above. Then*

$$\frac{\partial E(f_t)}{\partial t}\Big|_{t=0} = 0. \tag{4.7}$$

Proof. By proposition 4.1.3 and (4.6) we have

$$\frac{\partial E(f_t)}{\partial t}\Big|_{t=0} = \int_X \langle \omega, \beta \rangle = i \int \langle \omega, (D'' - D')\gamma \rangle.$$

As already noted in the discussion preceding lemma 1.8.5, being d-harmonic is equivalent to being D'-harmonic or D''-harmonic, hence $D'^*\omega = D''^*\omega = 0$, that gives the result. □

4.3 Variation of the energy under the $\mathbb{C}^*$-action

Let now X be a smooth Kähler manifold. In this section we want to investigate how the energy behaves under the $\mathbb{C}^*$-action on Higgs bundles given by (1.18). We will obtain from this discussion the converse to corollary 4.2.7, namely, that the only extremal points of the energy are $\mathbb{C}$-VHS.

Remark 4.3.1. the action of $S^1 \subset \mathbb{C}^*$ on a Higgs bundle $(\mathcal{E}, \theta)$ leaves the energy constant: As Simpson points out, if $t \in S^1$, then the harmonic metric K for $(\mathcal{E}, \theta)$ will still be harmonic for $(\mathcal{E}, t\theta)$, since harmonicity is equivalent to the vanishing of the curvature $F = \bar{\partial}\partial_K + \partial_K\bar{\partial} + [\theta, \theta_K^*]$ of the connection corresponding to $(\mathcal{E}, \theta)$, and the connection corresponding to $(\mathcal{E}, t\theta)$ under the same metric K has curvature $\bar{\partial}\partial_K + \partial_K\bar{\partial} + [t\theta, \bar{t}\theta_K^*] = F$, when $t \in S^1$. Thus from now on we shall assume t to be real.

Fix a smooth family of harmonic metrics f_t for $(\mathcal{E}, t\theta)$, so that we get a corrisponding family of flat bundles, whose monodromy representations we denote by $\rho_t \colon \Gamma \to G$. To compute the variation of the energy by means of proposition 4.1.3, then, we have to identify the harmonic representative ω of the cohomology class of $c(\gamma) = \frac{\partial \rho_t(\gamma)}{\partial t}\big|_{t=1} \cdot \rho_1(\gamma)^{-1} \in H^1(\Gamma, \mathrm{ad}(\rho_1))$. In the following, we denote by $(\mathbb{V}, \sigma, S)$ the $\mathbb{C}$-phls on $\mathcal{V} \to X$ induced by pull-back of the adjoint phls as in corollary 1.3.3. Recall that, by fact 1.3.4, this phls coincides with the one on $\mathrm{End}(\mathcal{E})$ induced by $(\mathcal{E}, \theta)$ and f_0.

Proposition 4.3.2. *Let ω be the harmonic representative of the first order variation of Higgs bundles $(\mathcal{E}, t\theta)$. Then:*

$$\frac{\partial E(\mathcal{E}, t\theta)}{\partial t}\Big|_{t=1} = \int_X \|\omega\|^2 \mathrm{dVol}_g \geqslant 0.$$

Proof. Since the family of Higgs bundles is given by $(\mathcal{E}, t\theta)$, its first order deformation is $B = \theta$; hence, using the isomorphism of Dolbeault cohomology and harmonic forms, as in lemma 1.8.8, we can write the harmonic 1-form ω as

$$\omega = \theta + D''\eta = \theta + \bar{\partial}\eta + [\theta, \eta], \tag{4.8}$$

for some section η of $\mathcal{V}$.

Thanks to proposition 4.1.3, and recalling that $\tilde{\beta} = \theta + \theta^*$, equation (4.8) allows us to write

$$\frac{\partial E(\mathcal{E}, t\theta)}{\partial t}\Big|_{t=1} = \int \langle \omega, \theta + \theta^* \rangle = \int \langle \omega, \theta \rangle + \langle \bar{\partial}\eta, \theta^* \rangle. \tag{4.9}$$

Notice that, using Stokes theorem and the fact that $\bar{\partial}\theta = 0$, which implies $0 = \Lambda\partial(\theta^*) = -2i \sum_j \partial_j \theta^*(\bar{\partial}_j)$,

$$\int \big\langle \bar{\partial}\eta, \theta^* \big\rangle = \sum_j \int \bar{\partial}_j \big\langle \eta, \theta^*(\bar{\partial}_j) \big\rangle - \int \big\langle \eta, \partial_j \theta^*(\bar{\partial}_j) \big\rangle = 0.$$

The result follows at once recalling that $\omega = \theta + D''\eta$, so that

$$\frac{\partial E(\mathcal{E}, t\theta)}{\partial t}\Big|_{t=1} = \int \langle \omega, \theta \rangle = \int \|\omega\|^2 - \langle \omega, D''\eta \rangle = \int \|\omega\|^2,$$

since $D''^*\omega = 0$. □

Example 4.3.3. Thanks to Simpson's theorem 1.9.6, we know that $[\mathcal{E}, \theta]$ comes from a VHS if and only if $(\mathcal{E}, \theta)$ is isomorphic to $(\mathcal{E}, t\theta)$ for every $t \in \mathbb{C}^*$. In particular, in this case the energy functional must be constant, so the derivative of proposition 4.3.2 must vanish. Recall from section 4 that, when $(\mathcal{E}, \theta)$ comes from a VHS, we have

$$\theta = i[\theta, \gamma] = i\big(\bar{\partial} + \mathrm{ad}(\theta)\big)(\gamma).$$

In particular, the Dolbeault 1-cohomology class of θ is null, so $\omega = 0$.

Lemma 4.3.4. *Let $(\mathcal{E}, \theta)$ be a Higgs bundle. Then it is a $\mathbb{C}$-VHS if, and only if, the 1-cohomology class*

$$\{\theta\} \in H^1_{\mathrm{Dol}}\big(X, \mathrm{End}(\mathcal{E})\big)$$

vanishes, i.e., $\theta = D''(\eta)$ for some $\eta \in \mathrm{End}(\mathcal{E})$.

Proof. The equivalence between θ vanishing in cohomology and being in the image of D'' is the last isomorphism of lemma 1.8.8.

On the one hand, if θ is a $\mathbb{C}$-VHS, we only have to apply lemma 4.2.5 and take $\eta = i\gamma$. On the other hand, if $\theta = D''(\eta)$, as $D'' = \bar{\partial} + \mathrm{ad}(\theta)$ and θ being of type $(1,0)$, we must have $\theta = [\theta, \eta]$. Define a group of automorphisms of $\mathcal{E}$ by $g_t = \exp(t\eta)$. Then a simple computation proves:

$$\mathrm{Ad}_{g_t}(\eta) = e^{-t}\theta.$$

This implies that the automorphism $g_t \in \mathrm{Aut}(\mathcal{E})$ sends θ to $e^t\theta$, hence $(\mathcal{E}, \theta)$ must be a $\mathbb{C}$-VHS. ☐

Definition 4.3.5. We say that the energy functional E has a critical point at a conjugacy class of representations $\{\rho_0\} \in \mathbb{M}_{\mathrm{B}}(X, G)$ if for every $\omega \in \mathcal{H}^1(X, \mathrm{Ad}(\rho_0))$

$$\int_X \langle \omega, \beta \rangle = 0.$$

By the results of chapter 2, this definition coincides with the usual one at least over Zariski-dense representations.

Corollary 4.3.6. *The critical points of the energy functional on $\mathbb{M}_{\mathrm{B}}(X, G)$ are exactly the representations coming from complex variations of Hodge structures.*

Proof. The fact that all $\mathbb{C}$-VHS are extrema of the energy is the content of corollary 4.2.7. Conversely, suppose that $(\mathcal{E}, \theta)$ is a Higgs bundle corresponding to a conjugacy class of representations $\{\rho_0\}$ which is a critical point of the energy. Letting ω be as in proposition 4.3.2, the hypothesis plus the mentioned proposition imply $\omega = 0$, that is, $\{\theta\} = 0$ in $H^1_{\mathrm{Dol}}(X, \mathrm{End}(\mathcal{E}))$. By lemma 4.3.4, then, $(\mathcal{E}, \theta)$ must be a $\mathbb{C}$-VHS. ☐

We also get the following restatement:

Corollary 4.3.7. *The gradient of the energy functional, defined with respect to the Weil-Petersson metric, is (twice) the vector field tangent to the $\mathbb{C}^*$-action*

Proof. This is a actually a consequence of the proof. Let χ denote the vector field tangent to the $\mathbb{C}^*$-action, and $\frac{\partial}{\partial t}\big|_{t=0}$ be a tangent direction to $\mathbb{R}_{\mathrm{B}}(X, G)$. Denoting by ω_χ and ω the two respective harmonic representatives, by definition of the Weil-Petersson metric, we have to prove that:

$$2\int_X \langle \omega_\chi, \omega \rangle = \int_X \langle \mathrm{grad}(E), \omega \rangle = \frac{\partial E}{\partial t}\Big|_{t=0}.$$

As above, we know that $\omega_\chi = \theta + D''\eta$, and by proposition 4.1.3 the right hand side equals $\int_X \langle \beta, \omega \rangle$. By harmonicity, $\int_X \langle D''\eta, \omega \rangle = 0$, hence:

$$2 \int_X \left\langle \omega_\chi, \omega \right\rangle = 2 \int_X \left\langle \theta, \omega \right\rangle = \int_X \left\langle \theta + \theta^*, \omega \right\rangle = \int_X \left\langle \beta, \omega \right\rangle = \frac{\partial E}{\partial t}\bigg|_{t=0},$$

since here $\theta \in \mathcal{A}^1(\tilde{X}, \mathfrak{g} \otimes \mathbb{C})$ and $\omega \in \mathcal{A}^1(\tilde{X}, \mathfrak{g})$, so that $\langle \theta, \omega \rangle = \langle \theta^*, \omega \rangle$. □

Chapter 5

Second order harmonic maps

Introduction au chapitre

Dans ce chapitre, on propose une analyse infinitésimale au second ordre des déformations des applications harmoniques tordues. La discussion suit celle faite pour le premier ordre : toutes les définitions impliquées sont données dans la Section 5.1, où l'on précise les concepts des déformations au second ordre $\rho_t^{(2)}$ et w et de quand ce dernier est harmonique ou $\rho_t^{(2)}$-équivariante. Il s'agit toujours des définitions induites par le cas d'une "vraie" déformation le long d'un groupe à un paramètre de représentations, en posant $w = \frac{D}{\partial t}\frac{\partial f_t}{\partial t}\big|_{t=0}$, mais les calculs dans ce cas sont plus lourdes; pour cette raison, ils sont reportés à la section 5.8. On donne aussi des définitions pour un couple de fonctions $(F, F_2)\colon \tilde{M} \to \mathfrak{g} \times \mathfrak{g}$ "de type harmonique et $\rho_t^{(2)}$-équivariantes", qui jouent le même rôle que la primitive F de ω dans le chapitre 3. Par contre, les définitions dans ce cas sont moins évidentes que dans l'analyse au premier ordre, ce qui reflète l'existence d'obstructions. Néanmoins, on donne une application ϑ_{J^2N} qui associe à chaque (F, F_2) de type harmonique et $\rho_t^{(2)}$-équivariante une déformation au second ordre w harmonique et $\rho_t^{(2)}$-équivariante.

Grâce à l'étude de l'action du centralisateur $H = Z_G(\mathrm{Image}(\rho_0))$ sur $H^1(M, \mathrm{Ad}(\rho_0))$, on arrive à donner des conditions nécessaires et suffisantes pour l'existence de (F, F_2). Tout d'abord, cette existence ne dépend que du premier ordre de la représentation, $\rho_t^{(1)}$. Deuxièmement, donnée (F, F_2) on peut définir une 1-forme ψ à valeurs dans $\mathrm{Ad}(\rho_0)$ par $\psi = \mathrm{d}F_2 + [\omega, F]$. Cette forme vérifie alors:

$$\begin{aligned} \mathrm{d}\psi &= -[\omega, \omega]; \\ \mathrm{d}^*\psi &= -\omega^*(\omega); \end{aligned}$$

où $\omega^*(\omega)$ est la section de $\mathrm{Ad}(\rho_0)$ définie en termes d'un système local orthonormé $\{E_j\}$ par $\omega^*(\omega) = \sum_j [\omega(E_j)^*, \omega(E_j)]$, le $*$ indiquant l'adjointion. L'existence d'une solution aux équations pour ψ est alors une condition nécessaire pour l'existence de (F, F_2); on démontre qu'elle est aussi suffisante. Notons que la solubilité de la première équation est en fait immédiate: l'existence d'une déformation au second ordre $\rho_t^{(2)}$ de $\rho_t^{(1)}$ est équivalente à l'annulation de la classe de cohomologie de $[\omega, \omega]$ (cfr. [GM87]). Par conséquent, une autre condition équivalente à l'existence de (F, F_2) est que $\omega^*(\omega)$ soit dans l'image de d^*; on peut reformuler cette condition en demandant que ω soit un point critique de la norme L^2 dans son orbite sous H. Dans le cas où G est un groupe complexe, la théorie des applications moments nous permet de conclure que ceci est équivalent à minimiser la norme L^2. Toujours dans le cas complexe, on arrive à démontrer que ceci est aussi équivalent à l'existence de *deux* déformations harmoniques, l'une (ρ_0, c)-équivariante et l'autre (ρ_0, ic)-équivariante. Ceci suggère que la bonne notion de déformabilité à considérer est celle le long d'une droite tangente complexe, et non pas réelle (par ailleurs, nous ne sommes pas arrivés à donner un exemple d'application déformable le long d'une direction réelle (ρ_0, c) mais pas de (ρ_0, ic)).

Ensuite, on donne des exemples explicites de déformations au premier ordre $\rho_t^{(1)}$ telles qu'aucune application harmonique ρ_0-équivairante f n'admet aucune déformation au second ordre harmonique et $\rho_t^{(2)}$-équivariante, pour n'importe quelle déformation au second ordre $\rho_t^{(2)}$ de $\rho_t^{(1)}$ (en effet, comme pour le (F, F_2), en toute généralité la déformabilité de f ne dépend que du premier ordre de la représentation). En accord avec le théorème de Corlette, on trouve de tels exemples en déformant la représentation triviale par des représentations unipotentes. Finalement, on observe que, si G est un groupe complexe, la déformabilité au second ordre de toutes métriques f (le long de la droite tangente complexe engendrée par $\rho_t^{(1)}$) est équivalente à la platitude de $H^1(M, \mathrm{Ad}(\rho_t^{(1)}))$ en tant que $\mathbb{R}[t]/(t^2)$-module.

5.1 Definitions

We start by the definition of second order deformations (f, v, w) of a smooth map $f\colon \tilde{M} \to N$ with values in a symmetric space, as well as second order deformations $\rho_t^{(2)} = (\rho_0, c, k)$ of a representation $\rho_0\colon \Gamma \to G$ in terms of the spaces J^2G and J^2N of 2-jets of G and N, respectively. Then we proceed to defining, for the triples (f, v, w) and for maps $(F, F_2)\colon \tilde{M} \to \mathfrak{g} \times \mathfrak{g}$, the concepts of $\rho_t^{(2)}$-equivariance and harmonicity.

In the following, we will make free use of the following isomorphisms: Recall that G is the Lie group of the real points of a connected reductive algebraic group $\mathbb{G}$. The space of 2-jets of G is identified as follows:

$$J^2G \overset{\text{def}}{=} \mathbb{G}\big(\mathbb{R}[t]/(t^3)\big) \longleftrightarrow G \times \mathfrak{g} \times \mathfrak{g}$$

where the bijection is the "right trivialization" making an infinitesimal curve $g(t)$ correspond to $(g, \xi, \mu) \in G \times \mathfrak{g} \times \mathfrak{g}$ defined by

$$g = g(0); \quad \xi = \frac{\partial g(t)}{\partial t}\Big|_{t=0} \cdot g^{-1}; \quad \mu = \frac{\partial}{\partial t}\Big(\frac{\partial g(t)}{\partial t} \cdot g(t^{-1})\Big)\Big|_{t=0}.$$

This bijection gives $G \times \mathfrak{g} \times \mathfrak{g}$ the following group structure (cfr. [Ber08], §23 and §24):

$$(g, \xi, \mu) \cdot (h, \eta, \nu) = \big(gh, \xi + \mathrm{Ad}_g(\eta), \mu + \mathrm{Ad}_g(\nu) + [\xi, \mathrm{Ad}_g(\eta)]\big). \tag{5.1}$$

The quotient J^2G/K identifies naturally, via this bijection, to $N \times \mathfrak{g} \times \mathfrak{g}$ (even as a bundle, the trivial bundle $N \times \mathfrak{g} \times \mathfrak{g}$ identifies with $J^2G \times_G N$). The space of 2-jets of N is defined accordingly, and as in remark 1.4.4, connectedness implies

$$J^2N = \mathbf{N}\big(\mathbb{R}[t]/(t^3)\big)^\circ = J^2G/J^2K.$$

We shall need a splitting of J^2N, induced by the canonical connection on N. This gives an isomorphism of bundles

$$J^2N \cong TN \times_N TN$$

by mapping a second order infinitesimal curve $n(t)$ to

$$(v, w) \overset{\text{def}}{=} \Big(\frac{\partial n(t)}{\partial t}\Big|_{t=0}, \frac{D}{\partial t}\frac{\partial n(t)}{\partial t}\Big|_{t=0}\Big).$$

With these premises, the following definitions are natural:

Definition 5.1.1. Let N be a symmetric space, $f\colon \tilde{M} \to N$ a smooth map. A second order deformation of f is a triple:

$$\Big(f,\ v \overset{\text{not}}{=} \frac{\partial f_t}{\partial t}\Big|_{t=0},\ w \overset{\text{not}}{=} \frac{D}{\partial t}\frac{\partial f_t}{\partial t}\Big|_{t=0}\Big), \tag{5.2}$$

where v and w are two sections of f^*TN. Equivalently, a second order deformation of f is a section of f^*J^2N.

Definition 5.1.2. A second order deformation (v, w) of a harmonic map $f\colon \tilde{M} \to N$ is said to be harmonic (to the second order) if, in terms of a local orthonormal frame $\{E_j\}$ of $\tilde{M}$,

$$\mathcal{J}(v) = 0; \quad \mathcal{J}(w) = 4 \sum_j R^N\big(\mathrm{d}f(E_j), v\big) \overset{N}{\nabla}_{E_j} v.$$

The fact that if f_t is a smooth family of harmonic maps, (v, w) defined as in (5.2) is harmonic to the second order is proved in section 5.8.

Definition 5.1.3. A representation $\rho_t^{(2)} = (\rho_0, c, k)\colon \Gamma \to J^2G \cong G \times \mathfrak{g} \times \mathfrak{g}$ will be called a second order deformation of the representation $\rho_0\colon \Gamma \to G$.

Notation 5.1.4. Given a second order deformation $\rho_t^{(2)} = (\rho_0, c, k)$, we will always denote by $\rho_t^{(1)} = (\rho_0, c)\colon \Gamma \to TG$ the first order order deformation of ρ_0 it determines.

Lemma 5.1.5. *Let $\rho_t^{(1)} = (\rho_0, c)\colon \Gamma \to TG$. The data of a second order deformation $\rho_t^{(2)}$ of $\rho_t^{(1)}$ is equivalent to that of a map $k\colon \Gamma \to \mathfrak{g}$ such that $(c, k) \in Z^1(\Gamma, \mathfrak{g}\times\mathfrak{g})$ is a 1-cocycle for the adjoint action of Γ on $\mathfrak{g}\otimes\mathbb{R}[t]/(t^2) \cong \mathfrak{g} \times \mathfrak{g}$, which is given by*

$$\gamma \cdot (\xi + t\mu) = \mathrm{Ad}_{\rho_0(\gamma)}(\xi) + t\Big(\mathrm{Ad}_{\rho_0(\gamma)}(\mu) + \big[c(\gamma), \mathrm{Ad}_{\rho_0(\gamma)}(\xi)\big]\Big). \tag{5.3}$$

Proof. Let $\rho_t^{(2)} = (\rho_0, c, k)$ be a second order deformation of ρ_0. The product law (5.1) implies:

$$\rho_t^{(2)}(\gamma\eta) = \Big(\rho_0(\gamma\eta), c(\gamma)+\mathrm{Ad}_{\rho_0(\gamma)}c(\eta), k(\gamma)+\mathrm{Ad}_{\rho_0(\gamma)}k(\eta)+\big[c(\gamma), \mathrm{Ad}_{\rho_0(\gamma)}c(\eta)\big]\Big).$$

One sees at once that this coincides with the expression given by the cocycle law, that in terms of our action (5.3), becomes:

$$\begin{aligned}(c+tk)(\gamma\eta) &= c(\gamma) + tk(\gamma) + \gamma \cdot \Big(c(\eta) + tk(\eta)\Big)\\ &= \Big(c(\gamma) + \mathrm{Ad}_{\rho_0(\gamma)}(\eta)\Big) + t\Big(k(\gamma) + \mathrm{Ad}_{\rho_0(\gamma)}k(\eta) + \big[c(\gamma), \mathrm{Ad}_{\rho_0(\gamma)}c(\eta)\big]\Big).\end{aligned}$$

□

Definition 5.1.6. Let $\rho_t^{(2)}\colon \Gamma \to J^2G$ be a second order deformation of a representation. A second order deformation (v, w) of a ρ_0-equivariant map $f\colon \tilde{M} \to N$ is called $\rho_t^{(2)}$-equivariant if:

$$\begin{aligned}f(\gamma\tilde{x}) &= \rho_0(\gamma) \cdot f(\tilde{x});\\ v(\gamma\tilde{x}) &= \rho_0(\gamma)_* v(\tilde{x}) + \vartheta_{TN}\big(f(\gamma\tilde{x}), c(\gamma)\big);\\ w(\gamma\tilde{x}) &= \rho_0(\gamma)_* w(\tilde{x})\\ &\qquad + \vartheta_{TN}\Big(f(\gamma\tilde{x}), k(\gamma) + 2\big[c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}, \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v(\tilde{x}))\big] + \big[c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}, c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\big]\Big).\end{aligned}$$

Recall that by $c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}$ we mean the projection of $c(\gamma) \in \mathfrak{g}$ onto $[\mathfrak{k}]_{\gamma\tilde{x}} = \mathrm{Ad}_{f(\gamma\tilde{x})}(\mathfrak{k})$, and similarly for $[\mathfrak{p}]$. Again, these equations will be proved when (v, w) are constructed from a smooth ρ_t-equivariant family f_t in section 5.8.

We now pass to the analysis of an application $(F, F_2)\colon \tilde{M} \to \mathfrak{g} \times \mathfrak{g}$. As we did for F in section 5, the idea is to give a definition of equivariance and harmonicity for these applications, too, and construct a correspondence sending a pair (F, F_2) to a second order deformation (v, w).

Definition 5.1.7. Let $\rho_t^{(2)}$ be a second order deformation of ρ_0. An application $(F, F_2)\colon \tilde{M} \to \mathfrak{g} \times \mathfrak{g}$ will be called $\rho_t^{(2)}$-equivariant if

$$\begin{aligned}(F(\gamma\tilde{x}), F_2(\gamma\tilde{x})) = \big(&\mathrm{Ad}_{\rho(\gamma)}(F(\tilde{x})) + c(\gamma)\\ &\mathrm{Ad}_{\rho(\gamma)}(F_2(\tilde{x})) + [c(\gamma), \mathrm{Ad}_{\rho(\gamma)}(F(\tilde{x}))] + k(\gamma)\big).\end{aligned}$$

This is clearly equivalent to asking that F be equivariant with respect to the representation $\rho_t^{(2)}$ and the natural action

$$J^2G \cong G \times \mathfrak{g} \times \mathfrak{g} \circlearrowright N \times \mathfrak{g} \times \mathfrak{g} \cong (G \times \mathfrak{g} \times \mathfrak{g})/K \cong J^2G/K \cong N \times_G J^2G.$$

Definition 5.1.8. Let $\rho_t^{(1)} = (\rho_0, c)$ be a first order deformation of ρ_0 and $\omega \in \mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$ the harmonic representative of the 1-cohomology class $\{c\}$, with respect to a metric $f\colon \tilde{M} \to N$. Let $\tilde{\omega}$ be its pull-back to $\tilde{M}$. We define the operators $D_2\colon \mathcal{A}^p_{\tilde{M}}(\mathfrak{g} \times \mathfrak{g}) \to \mathcal{A}^{p+1}_{\tilde{M}}(\mathfrak{g} \times \mathfrak{g})$ and $D_{2,*}\colon \mathcal{A}^1_{\tilde{M}}(\mathfrak{g} \times \mathfrak{g}) \to \mathcal{A}^0_{\tilde{M}}(\mathfrak{g} \times \mathfrak{g})$ by

$$D_2 = \begin{pmatrix} \mathrm{d} & \\ \mathrm{ad}(\tilde{\omega}) & \mathrm{d} \end{pmatrix}, \quad D_{2,*} = \begin{pmatrix} \mathrm{d}^* & \\ \tilde{\omega}^* & \mathrm{d}^* \end{pmatrix},$$

where we define $\tilde{\omega}^*\colon \mathcal{A}^1_{\tilde{M}}(\mathfrak{g}) \to \mathcal{C}^\infty(\tilde{M}, \mathfrak{g})$ sending $\tilde{\alpha}$ to the contraction of $\mathrm{ad}(\tilde{\omega}^*)(\tilde{\alpha}) = -\mathrm{ad}(\sigma(\tilde{\omega}))(\tilde{\alpha})$, that is, in terms of a local orthonormal frame $\{E_j\}$:

$$\tilde{\omega}^*(\tilde{\alpha}) = \sum_j \big[\tilde{\omega}(E_j)^{[\mathfrak{p}]} - \tilde{\omega}(E_j)^{[\mathfrak{k}]}, \tilde{\alpha}(E_j)\big].$$

Remark 5.1.9. The D_2 thus defined induces a flat connection on $\tilde{M} \times \mathfrak{g} \times \mathfrak{g}$, which is Γ-invariant for the adjoint action of Γ on both factors $\mathfrak{g}$, hence it induces one on $\mathcal{V} \oplus \mathcal{V}$, where $\mathcal{V} = \tilde{M} \times_\Gamma \mathfrak{g}$ is the flat bundle underlying the pull-back of the adjoint phls on N. The definition of $\tilde{\omega}^*$ is given in a way as to induce a $\omega^*\colon \mathcal{A}^1(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{V})$ such that for every section ξ of $\mathcal{V}$ and for every $\mathcal{V}$-valued 1-form α, we have

$$\big\langle [\omega, \xi], \alpha \big\rangle = \big\langle \xi, \omega^*(\alpha) \big\rangle.$$

Definition 5.1.10. An application $(F, F_2)\colon \tilde{M} \to \mathfrak{g} \times \mathfrak{g}$ is said to be of "harmonic type" (relatively to ω) if

$$D_{2,*} D_2 \begin{pmatrix} F \\ F_2 \end{pmatrix} = 0.$$

Definition 5.1.11. The correspondence $\vartheta_{J^2N}\colon N \times \mathfrak{g} \times \mathfrak{g} \to TN \times_N TN$ is defined (in terms of $\vartheta_{TN}\colon N \times \mathfrak{g} \to TN$ as in section 1.4) by:

$$\vartheta_{J^2N}(f, F, F_2)_{\tilde{x}} = \Big(f, v = \vartheta_{TN}(F(\tilde{x})), w = \vartheta_{TN}\big(F_2 + [F^{[\mathfrak{k}]}(\tilde{x}), F^{[\mathfrak{p}]}(\tilde{x})]\big)\Big)$$

(here, as usual, $F^{[\mathfrak{p}]}(\tilde{x})$ equals the projection of $F(\tilde{x})$ on $[\mathfrak{p}]_{\tilde{x}} = \mathrm{Ad}_{f(\tilde{x})}(\mathfrak{p})$).

Lemma 5.1.12. *The following diagram commutes:*

$$\begin{array}{ccc} J^2G & \xrightarrow{\ r\ } & G \times \mathfrak{g} \times \mathfrak{g} \\ \Big\downarrow{\scriptstyle \pi_{N,*}} & & \Big\downarrow{\scriptstyle /K} \\ & & N \times \mathfrak{g} \times \mathfrak{g} \\ & & \Big\downarrow{\scriptstyle \vartheta_{J^2N}} \\ J^2N & \xrightarrow{\ \sim\ } & TN \times_N TN \end{array}$$

Proof. Take a $n(t) = \pi_{N,*} \circ r^{-1}(g, \xi, \mu)$. We want to prove that the image of $n(t)$ through the trivialization induced by the canonical connection is the same as $\vartheta_{J^2N}(g, \xi, \mu) = \big(\vartheta_{TN}(gK, \xi), \vartheta_{TN}\big(gK, \mu + [\xi^{[\mathfrak{k}]}, \xi^{[\mathfrak{p}]}]\big)\big)$. Recall that the canonical connection is induced by $\mathrm{d}^{\mathrm{can}} = \mathrm{d} - [\beta, \cdot]$. Thus, the image of $n(t)$ is

$$(v, w) = \Big(\vartheta_{TN}(gK, \xi), \vartheta_{TN}\big(gK, \mu - \big[\beta(\frac{\partial}{\partial t}), \xi\big]\big)\Big).$$

Now $\beta(\frac{\partial}{\partial t}) = \beta_N(v) = \xi^{[\mathfrak{p}]}$ by definition, so that

$$w = \vartheta_{TN}\Big(gK, \mu - \big[\xi^{[\mathfrak{p}]}, \xi\big]\Big) = \vartheta_{TN}\Big(gK, \mu - \big[\xi^{[\mathfrak{p}]}, \xi^{[\mathfrak{k}]}\big]\Big),$$

since ϑ_{TN} keeps the projection on $[\mathfrak{p}]$ only. This last expression equals the second term of $\vartheta_{J^2N}(gK, \xi, \mu)$. □

In section 5.2 we will prove that, given (F, F_2) of harmonic type and $\rho_t^{(2)}$-equivariant and defining $(v, w) = \vartheta_{J^2N}(F, F_2)$ we obtain a harmonic and $\rho_t^{(2)}$-equivariant second order deformation of f. Thus, to construct second order deformations of a metric f it is enough to construct a $\rho_t^{(2)}$-equivariant (F, F_2) of harmonic type.

5.2 The correspondence ϑ_{J^2N}

In this section, $\rho_t^{(2)} = (\rho_0, c, k)\colon \Gamma \to J^2G$ will be a second order deformation of ρ_0, and $f\colon \tilde{M} \to N$ will be ρ_0-equivariant and harmonic.

Lemma 5.2.1. *Let $(F, F_2)\colon \tilde{M} \to \mathfrak{g} \times \mathfrak{g}$ be $\rho_t^{(2)}$-equivariant. Define*

$$(v, w) = \vartheta_{J^2N}(f, F, F_2). \tag{5.4}$$

Then (f, v, w) is equivariant, in the sense of definition 5.1.6.

Proof. The equivariance of v has already been proved in section 3. As for w, recalling that $(\mathrm{Ad}_{\rho_0(\gamma)}(\xi))^{[\mathfrak{p}]}_{\gamma\tilde{x}} = \mathrm{Ad}_{\rho_0(\gamma)}(\xi^{[\mathfrak{p}]}_{\tilde{x}})$, where $\xi\colon \tilde{M} \to \mathfrak{g}$ and $\xi^{[\mathfrak{p}]}_{\tilde{x}}$ denotes the projection of $\xi(\tilde{x})$ onto $\mathrm{Ad}_{f(\tilde{x})}(\mathfrak{p})$, the equivariance of (F, F_2) implies

$$\begin{aligned}
(F_2^{[\mathfrak{p}]}(\gamma\tilde{x})) &= \mathrm{Ad}_{\rho_0(\gamma)}\big(F_2^{[\mathfrak{p}]}(\tilde{x})\big) + \big[c(\gamma), \mathrm{Ad}_{\rho_0(\gamma)}(F(\tilde{x}))\big]^{[\mathfrak{p}]}_{\gamma\tilde{x}} + k(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}} \\
&= \mathrm{Ad}_{\rho_0(\gamma)}\big(F_2^{[\mathfrak{p}]}(\tilde{x})\big) + \big[c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}, \mathrm{Ad}_{\rho_0(\gamma)}F^{[\mathfrak{k}]}(\tilde{x})\big] \\
&\quad + \big[c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}, \mathrm{Ad}_{\rho_0(\gamma)}F^{[\mathfrak{p}]}(\tilde{x})\big] + k(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}, \\
\big[F^{[\mathfrak{k}]}(\gamma\tilde{x}), F^{[\mathfrak{p}]}(\gamma\tilde{x})\big] &= \big[c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}, c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\big] + \mathrm{Ad}_{\rho_0(\gamma)}\big[F^{[\mathfrak{k}]}(\tilde{x}), F^{[\mathfrak{p}]}(\tilde{x})\big] \\
&\quad + \big[c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}, \mathrm{Ad}_{\rho_0(\gamma)}F^{[\mathfrak{p}]}(\tilde{x})\big] - \big[c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}, \mathrm{Ad}_{\rho_0(\gamma)}F^{[\mathfrak{k}]}(\tilde{x})\big].
\end{aligned}$$

Adding these expressions together, we get

$$\begin{aligned}
\big(F_2^{[\mathfrak{p}]} + [F^{[\mathfrak{k}]}, F^{[\mathfrak{p}]}]\big)(\gamma\tilde{x}) &= \mathrm{Ad}_{\rho_0(\gamma)}\big(F_2^{[\mathfrak{p}]}(\tilde{x}) + [F^{[\mathfrak{k}]}, F^{[\mathfrak{p}]}](\tilde{x})\big) + k(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}} \\
&\quad + 2\big[c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}, \mathrm{Ad}_{\rho_0(\gamma)}F^{[\mathfrak{p}]}(\tilde{x})\big] + \big[c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}, c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\big].
\end{aligned}$$

If (v, w) is $\rho_t^{(2)}$-equivariant, then $\beta_N(w)$ has exactly the same kind of equivariance. □

Lemma 5.2.2. *Let $\xi, \eta\colon \tilde{M} \to \mathfrak{g}$ be functions. Then, in terms of a local orthonormal frame $\{E_j\}$,*

$$J\big([\xi, \eta]\big) = \big[J(\xi), \eta\big] + \big[\xi, J(\eta)\big] + 2\sum_j \Big(\big[[\beta(E_j), \xi], [\beta(E_j), \eta]\big] - \big[\overset{\mathrm{can}}{\nabla}_{E_j}\xi, \overset{\mathrm{can}}{\nabla}_{E_j}\eta\big]\Big).$$

Proof. Recall from lemma 1.6.6 that $J = \mathrm{d}^*\mathrm{d}$, where $\mathrm{d}^*\alpha = -\sum_j \tilde{\nabla}_{E_j}\alpha(E_j)$, $\tilde{\nabla} = \overset{\mathrm{can}}{\nabla} - \mathrm{ad}(\beta)$ and $\mathrm{d} = \overset{\mathrm{can}}{\nabla} + \mathrm{ad}(\beta)$. Further, with the same proof as in lemma 1.6.7, one proves that $\tilde{\nabla}[\xi, \eta] = [\tilde{\nabla}\xi, \eta] + [\xi, \tilde{\nabla}\eta]$. Then one only has to compute

$$J\big([\xi,\eta]\big) = [J(\xi),\eta] + [\xi, J(\eta)] - \sum_j \Big(\big[\mathrm{d}_{E_j}\xi, \tilde{\nabla}_{E_j}\eta\big] + \big[\tilde{\nabla}_{E_j}\xi, \mathrm{d}_{E_j}\eta\big] \Big)$$
$$= [J(\xi),\eta] + [\xi, J(\eta)] - \sum_j \Big[\overset{\mathrm{can}}{\nabla}_{E_j}\xi + \big[\beta(E_j),\xi\big], \overset{\mathrm{can}}{\nabla}_{E_j}\eta - \big[\beta(E_j),\eta\big] \Big]$$
$$- \sum_j \Big[\overset{\mathrm{can}}{\nabla}_{E_j}\xi - \big[\beta(E_j),\xi\big], \overset{\mathrm{can}}{\nabla}_{E_j}\eta + \big[\beta(E_j),\eta\big] \Big],$$

and notice that every "mixed term" cancels out. □

Lemma 5.2.3. *Let (F, F_2) be of harmonic type relatively to ω, and suppose that $\mathrm{d}F = \omega$. Then (v, w) defined as in (5.4) is a second order harmonic deformation of f.*

Proof. Again, since the first component of the definition 5.1.10 gives $J(F) = 0$, the problem has already been solved for v in section 3, so we focus on computing $\mathcal{J}(w)$, or, equivalently, $J(\beta_N(w))$. Also recall that J respects the $[\mathfrak{p}] \oplus [\mathfrak{k}]$ decomposition, hence $J(F_2^{[\mathfrak{p}]}) = J(F_2)^{[\mathfrak{p}]}$.

Using $\mathrm{d}F = \omega$, the second component of the definition 5.1.10 gives an expression for $J(F_2)$:

$$\begin{pmatrix} \mathrm{d}^* & \\ \omega^* & \mathrm{d}^* \end{pmatrix} \begin{pmatrix} \mathrm{d}F \\ [\omega, F] + \mathrm{d}F_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega^*(\omega) - \sum_j \big[\omega(E_j), \tilde{\nabla}_{E_j}F\big] + J(F_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that is:

$$J(F_2) = \sum_j \big[\omega(E_j), \tilde{\nabla}_{E_j}F\big] - \omega^*(\omega). \tag{5.5}$$

Start by analyzing the second summand $-\omega^*(\omega)$ of (5.5). Writing $\omega = \omega^{[\mathfrak{k}]} + \omega^{[\mathfrak{p}]}$, this equals

$$-\omega^*(\omega) = \sum_j \big[\omega^{[\mathfrak{k}]}(E_j) - \omega^{[\mathfrak{p}]}(E_j), \omega(E_j)\big] = 2\sum_j \big[\omega^{[\mathfrak{k}]}(E_j), \omega^{[\mathfrak{p}]}(E_j)\big] \in [\mathfrak{p}].$$

Using $\omega = \mathrm{d}F = \overset{\mathrm{can}}{\nabla} F + [\beta, F]$, we get:

$$-\omega^*(\omega) = 2\sum_j \Big[\overset{\mathrm{can}}{\nabla}_{E_j}F^{[\mathfrak{k}]} + [\beta(E_j), F^{[\mathfrak{p}]}], \overset{\mathrm{can}}{\nabla}_{E_j}F^{[\mathfrak{p}]} + [\beta(E_j), F^{[\mathfrak{k}]}] \Big]$$
$$= \sum_j 2\big[[\beta(E_j), F^{[\mathfrak{p}]}], \overset{\mathrm{can}}{\nabla}_{E_j}F^{[\mathfrak{p}]}\big] + 2\big[\overset{\mathrm{can}}{\nabla}_{E_j}F^{[\mathfrak{k}]}, \overset{\mathrm{can}}{\nabla}_{E_j}F^{[\mathfrak{p}]}\big]$$
$$- 2\Big[\big[\beta(E_j), F^{[\mathfrak{k}]}\big], \big[\beta(E_j), F^{[\mathfrak{p}]}\big]\Big] - 2\big[[\beta(E_j), F^{[\mathfrak{k}]}], \overset{\mathrm{can}}{\nabla}_{E_j}F^{[\mathfrak{k}]}\big]$$

Since $\omega(E_j) = \overset{\text{can}}{\nabla}_{E_j} F + [\beta(E_j), F]$ and $\tilde{\nabla}_{E_j} F = \overset{\text{can}}{\nabla}_{E_j} F - [\beta(E_j), F]$, one gets immediately that the first summand of (5.5) equals $2\big[[\beta(E_j), F], \overset{\text{can}}{\nabla}_{E_j} F\big]$. Its projection on $[\mathfrak{p}]$ is

$$2\big[[\beta(E_j), F], \overset{\text{can}}{\nabla}_{E_j} F\big]^{[\mathfrak{p}]} = 2\big[[\beta(E_j), F^{[\mathfrak{k}]}], \overset{\text{can}}{\nabla}_{E_j} F^{[\mathfrak{k}]}\big] + 2\big[[\beta(E_j), F^{[\mathfrak{p}]}], \overset{\text{can}}{\nabla}_{E_j} F^{[\mathfrak{p}]}\big].$$

Summing this expression to the one obtained for $-\omega^*(\omega)$,

$$\begin{aligned} J(F_2^{[\mathfrak{p}]}) = \sum_j & 4\big[[\beta(E_j), F^{[\mathfrak{p}]}], \overset{\text{can}}{\nabla}_{E_j} F^{[\mathfrak{p}]}\big] + 2\big[\overset{\text{can}}{\nabla}_{E_j} F^{[\mathfrak{k}]}, \overset{\text{can}}{\nabla}_{E_j} F^{[\mathfrak{p}]}\big] \\ & - 2\Big[\big[\beta(E_j), F^{[\mathfrak{k}]}\big], \big[\beta(E_j), F^{[\mathfrak{p}]}\big]\Big]. \end{aligned}$$

Now, since $J(F^{[\mathfrak{k}]}) = J(F^{[\mathfrak{p}]}) = 0$, lemma 5.2.2 gives

$$J\big([F^{[\mathfrak{k}]}, F^{[\mathfrak{p}]}]\big) = \sum_j 2\Big[\big[\beta(E_j), F^{[\mathfrak{k}]}\big], \big[\beta(E_j), F^{[\mathfrak{p}]}\big]\Big] - 2\big[\overset{\text{can}}{\nabla}_{E_j} F^{[\mathfrak{k}]}, \overset{\text{can}}{\nabla}_{E_j} F^{[\mathfrak{p}]}\big],$$

and adding the two expressions we find exactly

$$J(\beta_N(w)) = \sum_j 4\big[[\beta(E_j), F^{[\mathfrak{p}]}], \overset{\text{can}}{\nabla}_{E_j} F^{[\mathfrak{p}]}\big] = \beta_N\Big(\sum_j R^N(\mathrm{d}f(E_j), v)\overset{N}{\nabla}_{E_j} v\Big).$$

□

Corollary 5.2.4. *Let $(F, F_2)\colon \tilde{M} \to \mathfrak{g}\times\mathfrak{g}$ be $\rho_t^{(2)}$-equivariant and of harmonic type. Then (v, w) defined as in* (5.4) *is $\rho_t^{(2)}$-equivariant and harmonic.*

Proof. We only need to apply the two lemmas above, combining them with lemma 3.4.2, which grants that $\mathrm{d}F = \omega$. □

Remark 5.2.5. Thus, if (F, F_2) is $\rho_t^{(2)}$-equivariant and of harmonic type, the first component of

$$D_2 \begin{pmatrix} F \\ F_2 \end{pmatrix}$$

is uniquely determined. However, this needs not be true for the second one, as it will become clear in the next sections (cfr. corollary 5.4.3).

5.3 The action of H on $H^1(M, \mathrm{Ad}(\rho_0))$

Recall that in section 3 we defined $H = Z_G(\mathrm{Image}(\rho_0))$ and denoted by $\mathfrak{h}$ its Lie algebra. This group acts by conjugation on both $Z^1(M, \mathbb{V})$ and $Z^1(\Gamma, \mathrm{Ad}(\rho_0))$ (and those actions pass to cohomology, where they coincide). Fix a metric $f\colon \tilde{M} \to N$. Then $H^1(M, \mathbb{V}) \cong \mathcal{H}^1(M, \mathbb{V})$, the space of harmonic 1-forms. The group H acts on this space by adjunction as well:

Lemma 5.3.1. *The action of H on $Z^1(M, \mathbb{V})$ leaves the subspace $\mathcal{H}^1(M, \mathbb{V})$ invariant. In particular:*

$$\forall \xi \in \mathfrak{h},\ \omega \in \mathcal{H}^1(M, \mathbb{V}), \quad [\omega, \xi] \textit{ is harmonic.} \tag{5.6}$$

Proof. We want to prove that for every h in H, if ω is a harmonic form then also $\mathrm{Ad}_h(\omega)$ is. It is evidently d-closed. We start by proving d^*-closedness when $h \in H \cap K$, which we may assume to be the maximal compact subgroup of H. Observe also that $\mathrm{Ad}_h(\omega)$ is harmonic for the metric f if and only if ω is harmonic for $h^{-1}f$, since for every $\xi \in \mathcal{C}^\infty(\mathcal{V})$,

$$\left\langle \mathrm{Ad}_h(\omega), \mathrm{d}\xi \right\rangle_f = \left\langle \omega, \mathrm{Ad}_{h^{-1}}(\mathrm{d}\xi) \right\rangle_{h^{-1}f} = \left\langle \omega, \mathrm{d}\xi' \right\rangle_{h^{-1}f}, \quad \xi' = \mathrm{Ad}_{h^{-1}}\xi \in \mathcal{C}^\infty(\mathcal{V}).$$

Thus we may also suppose that $f = f_0\colon \tilde{M} \to G_0/K_0$, as in notation 1.2.5. Then the above can be expressed in terms of a section $s_0\colon \tilde{M} \to G_0$ of f_0 as

$$\left\langle \omega, \mathrm{d}\xi' \right\rangle_{h^{-1}f_0} = S(\omega, \mathrm{Ad}_{s_0^{-1}}\mathrm{Ad}_h \sigma_0 \mathrm{Ad}_{h^{-1}}\mathrm{Ad}_{s_0}\mathrm{d}\xi').$$

Now the flat Cartan involution σ_0 is the one fixing $\mathfrak{k} \oplus \mathfrak{p}$, hence it is $\mathrm{Ad}(K)$-invariant. This implies that

$$\left\langle \omega, \mathrm{d}\xi' \right\rangle_{h^{-1}f_0} = S(\omega, \mathrm{Ad}_{s_0^{-1}}\sigma_0 \mathrm{Ad}_{s_0}\mathrm{d}\xi') = \left\langle \omega, \mathrm{d}\xi' \right\rangle_{f_0} = 0.$$

To complete the proof, we only need to prove (5.6), since $H = H^\circ \cdot (H \cap K)$, where H° denotes the identity component of H, and the connected group H° leaves $\mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$ invariant if and only if its Lie algebra $\mathfrak{h}$ does. On the one hand, both ω and ξ are d-closed, so their bracket is, too. On the other hand, by lemma 1.6.3, we have, for an orthonormal basis E_j,

$$\mathrm{d}^*[\omega, \xi] = -\sum_j \tilde{\nabla}_{E_j}[\omega(E_j), \xi] = \sum_j -[\omega(E_j), \overset{\mathrm{can}}{\nabla}_{E_j}\xi] + \left[\omega(E_j), [\tilde{\beta}(E_j), \xi]\right],$$

where we have used that $-\sum \tilde{\nabla}_{E_j}\omega(E_j) = \mathrm{d}^*\omega = 0$ and that by definition $\tilde{\nabla}_X(\xi) = \overset{\mathrm{can}}{\nabla}_X(\xi) - [\tilde{\beta}(X), \xi]$. Then, as in the proof of corollary 1.6.9, one can integrate by parts the equality $0 = \langle J\xi, \xi \rangle$ to obtain both $\overset{\mathrm{can}}{\nabla}\xi = 0$ and $[\tilde{\beta}, \xi] = 0$, thus proving the claim. □

Notation 5.3.2. We denote by $\mathrm{Ad}(\rho_t^{(1)})$ the local system $\tilde{M} \times_\Gamma (\mathfrak{g} \otimes \mathbb{R}[t]/(t^2))$ associated to the action of Γ on $\mathfrak{g} \otimes \mathbb{R}[t]/(t^2)$ in (5.3).

We have an exact sequence of sheaves

$$0 \to \mathrm{Ad}(\rho_0) \xrightarrow{\times t} \mathrm{Ad}(\rho_t^{(1)}) \xrightarrow{\mathrm{mod}\ t} \mathrm{Ad}(\rho_0) \to 0. \tag{5.7}$$

Remark that $H^0(M, \mathrm{Ad}(\rho_0)) = \mathfrak{h}$.

Lemma 5.3.3. *Let $\mathfrak{h}' \subset \mathfrak{h}$ be the image of $H^0(M, \mathrm{Ad}(\rho_t^{(1)})) \to H^0(M, \mathrm{Ad}(\rho_0))$. Then*

$$\mathfrak{h}' = \{\xi \in \mathfrak{h} \ : \ [\omega, \xi] = 0\} \tag{5.8}$$

is the Lie algebra of H', the subgroup of H fixing $\{c\}$ (or, equivalently, ω).

Proof. The global sections of the bundle $\mathrm{Ad}(\rho_t^{(1)})$ are the $\xi + t\mu \in \mathfrak{g} \otimes \mathbb{R}[t]/(t^2)$ invariant under the action of Γ, that is:

$$\begin{cases} \mathrm{Ad}_{\rho(\gamma)}(\xi) = \xi; \\ \mathrm{Ad}_{\rho(\gamma)}(\mu) = \mu - [c(\gamma), \xi]. \end{cases} \tag{5.9}$$

In particular, the ξ's appearing in this expression are exactly the $\xi \in \mathfrak{h}$ such that $[c(\gamma), \xi] = \delta(\mu)(\gamma)$, where $\delta\colon C^0(\Gamma, \mathfrak{g}) \to Z^1(\Gamma, \mathfrak{g})$ denotes the coboundary of group cohomology. This means exactly $\xi \in \mathfrak{h}'$. □

5.4 Construction of F_2 and obstructions

Definition 5.4.1. Let (F, F_2) be $\rho_t^{(2)}$-equivariant and of harmonic type. Define the $\mathcal{V}$-valued 1-form $\psi = \psi(F, F_2)$ by

$$\begin{pmatrix} \omega \\ \psi \end{pmatrix} = D_2 \begin{pmatrix} F \\ F_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}F \\ \mathrm{d}F_2 + [\omega, F] \end{pmatrix}.$$

Since D_2 is flat and (F, F_2) is of harmonic type, as in definition 5.1.10, the 1-form ψ satisfies the following equations:

$$\begin{aligned} \mathrm{d}\psi &= -[\omega, \omega]; \\ \mathrm{d}^*\psi &= -\omega^*(\omega) = 2 \cdot \sum_i \left[\omega(E_i)^{[\mathfrak{k}]}, \omega(E_i)^{[\mathfrak{p}]}\right] \in \mathcal{C}^\infty(\mathcal{V}^-). \end{aligned} \tag{5.10}$$

In this section we will prove that the existence of a solution to (5.10) is not only necessary, but also sufficient, for the existence of a $\rho_t^{(2)}$-equivariant (F, F_2) of harmonic type. We prove that the existence of F_2 does not depend on the first order F chosen:

Lemma 5.4.2. *Let (F, F_2) be $\rho_t^{(2)}$-equivariant and of harmonic type, and let $F' = F + \xi$, with $\xi \in \mathfrak{h}$, be any other map which is $\rho_t^{(1)}$-equivariant and of harmonic type. Then*

$$(F', F_2') = \big(F + \xi, F_2 + [F, \xi]\big)$$

is $\rho_t^{(2)}$-equivariant and of harmonic type.

Proof. To compute equivariance:

$$\begin{aligned}
&F_2'(\gamma\tilde{x}) = F_2(\gamma\tilde{x}) + [F(\gamma\tilde{x}), \xi] \\
&= \mathrm{Ad}_{\rho_0(\gamma)} F_2(\tilde{x}) + \big[c(\gamma), \mathrm{Ad}_{\rho_0(\gamma)} F(\tilde{x})\big] + k(\gamma) + \mathrm{Ad}_{\rho_0(\gamma)}[F(\tilde{x}), \xi] + [c(\gamma), \xi] \\
&= \mathrm{Ad}_{\rho_0(\gamma)}\big(F_2(\tilde{x}) + [F(\tilde{x}), \xi]\big) + \big[c(\gamma), \mathrm{Ad}_{\rho_0(\gamma)} F'(\tilde{x})\big] + k(\gamma).
\end{aligned}$$

For the harmonicity, observe that

$$\begin{pmatrix} \omega' \\ \psi' \end{pmatrix} = D_2 \begin{pmatrix} F' \\ F_2' \end{pmatrix} = \begin{pmatrix} \omega \\ \mathrm{d}F_2' + [\omega, F'] \end{pmatrix} = \begin{pmatrix} \omega \\ \psi + 2[\omega, \xi] \end{pmatrix}. \tag{5.11}$$

Since (F, F_2) being of harmonic type is equivalent to ψ' satisfying (5.10), and since $2[\omega, \xi]$ is harmonic by lemma 5.3.1, we conclude. □

Corollary 5.4.3. *Given $\rho_t^{(2)}$ and f, the 1-form $\psi = \psi(F, F_2)$ is unique if, and only if, $\mathfrak{h} = \mathfrak{h}'$.*

Proof. Let (F, F_2) and (F', F_2') be two $\rho_t^{(2)}$-equivariant maps of harmonic type, so that $F' = F + \xi$. By the lemma, every $\xi \in \mathfrak{h}$ can occur. Then $F_2' - F_2 - [F, \xi]$ is $\mathrm{Ad}(\rho_0)$ equivariant and is killed by J, hence $F_2' = F_2 + [F, \xi] + \eta$, for some $\eta \in \mathfrak{h}$. In particular, the 1-form ψ' it defines is the one in (5.11), namely,

$$\psi' = \psi + 2[\omega, \xi].$$

Thus, $\psi' = \psi$ if and only if $\xi \in \mathfrak{h}'$. □

Now we study the problem of the existence of a ψ satisfying (5.10).

Fact 5.4.4. *If the representation $\rho_t^{(1)} = (\rho_0, c) \colon \Gamma \to TG$ can be extended to the second order, denoting by ω the harmonic representative of c, then the following 2-cohomology class vanishes:*

$$0 = \big\{[\omega, \omega]\big\} \in H^2(M, \mathrm{Ad}(\rho_0)).$$

Proof. Also see [GM88], §4.4. By hypothesis, $c \in Z^1(\mathrm{Ad}(\rho_0))$ can be extended to (c, k), that is, it is in the image of

$$\mathrm{mod}\ t\colon Z^1(\Gamma, \mathfrak{g} \otimes \mathbb{R}[t]/(t^2)) \to Z^1(\Gamma, \mathfrak{g}).$$

In particular, its cohomology class is in the image of the corresponding map. Since group cohomology equals the cohomology of local systems, we can interpret this fact in terms of the long exact cohomology sequence of (5.7), thus obtaining:

$$\to H^1(M, \mathrm{Ad}(\rho_t^{(1)})) \xrightarrow{\mathrm{mod}\ t} H^1(M, \mathrm{Ad}(\rho_0)) \xrightarrow{\delta} H^2(M, \mathrm{Ad}(\rho_0)) \to \dots$$

Then, $\{c\}$ being in the image of mod t forces $\delta(\{c\}) = \{c \cup c\} = 0$, hence $\{[\omega, \omega]\} = 0$. □

From now on, we suppose that $\rho_t^{(1)}$ can be extended to a second order deformation $\rho_t^{(2)}$. Hence, fact 5.4.4 grants the existence of some ψ_0 such that $\mathrm{d}\psi_0 = -[\omega, \omega]$. To construct ψ, we look for a section $\eta \in \mathcal{C}^\infty(\mathcal{V})$ such that $J(\eta) = \mathrm{d}^*\mathrm{d}\eta = -\omega^*(\omega) - \mathrm{d}^*\psi_0$.

Lemma 5.4.5. *The self adjoint operator* $J\colon \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{V})$ *determines an orthogonal decomposition*

$$\mathcal{C}^\infty(\mathcal{V}) = \mathfrak{h} \oplus \mathrm{Image}(J),$$

Further, this splitting is compatible with projections to $[\mathfrak{p}]$ *and* $[\mathfrak{k}]$.

Proof. This is essentially the Hodge theorem on the Riemannian manifold M with coefficients in the vector bundle $\mathcal{V}$. Indeed, J is an elliptic self-adjoint operator, hence the space of sections of $\mathcal{V}$ is the orthogonal sum of $\mathrm{Ker}(J)$ and $\mathrm{Image}(J)$ (see for example [Dem], chap. VI, Corollary 2.4). Now $\mathfrak{h} = \mathcal{C}^\infty(\mathcal{V}) \cap \mathrm{Ker}(\mathrm{d})$, so we have $\mathrm{Ker}(J) = \mathrm{Ker}(\mathrm{d}^*\mathrm{d}) \subseteq \mathrm{Ker}(\mathrm{d})$, but in fact we have equality, as an integration by parts shows.

Compatibility with $[\mathfrak{p}] \oplus [\mathfrak{k}]$ follows from the decomposition $\mathfrak{h} = \mathfrak{h}^{\mathfrak{p}} \oplus \mathfrak{h}^{\mathfrak{k}}$ of corollary 1.6.9 and corollary 1.6.8. □

With these results at hand, we can prove that solving (5.10) is equivalent to finding a $\rho_t^{(2)}$-equivariant (F, F_2) of harmonic type, and characterize when this is possible.

Proposition 5.4.6. *The following are equivalent:*

1. *The system* (5.10) *admits a solution;*

2. *The section $\omega^*(\omega)$ is orthogonal to $\mathfrak{h}$;*

3. *The 1-form ω is a critical point for $\int \|\omega\|^2$ in its orbit*

$$H \cdot \omega \subseteq \mathcal{H}^1(M, \mathrm{Ad}(\rho_0));$$

4. *There exists a pair (F, F_2) which is $\rho_t^{(2)}$-equivariant and of harmonic type;*

5. *For every $F \colon \tilde{M} \to \mathfrak{g}$ such that $\mathrm{d}F = \omega$ and $F(\gamma\tilde{x}) = \mathrm{Ad}_{\rho_0(\gamma)} F(\tilde{x}) + c(\gamma)$, there exists a $F_2 \colon \tilde{M} \to \mathfrak{g}$ such that (F, F_2) is $\rho_t^{(2)}$-equivariant and of harmonic type.*

Proof. (1) $\implies$ (2): If a ψ satisfying (5.10) exists, then $\mathrm{d}^*\psi = -\omega^*(\omega)$, hence $\langle \omega^*(\omega), \xi \rangle = -\langle \psi, \mathrm{d}\xi \rangle = 0$ for all $\xi \in \mathfrak{h}$.

(2) $\iff$ (3): This follows at once from:

$$0 = \big\langle \omega^*(\omega), \xi \big\rangle = \big\langle \omega, [\omega, \xi] \big\rangle = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big\| \mathrm{Ad}_{\exp(t\xi)}(\omega) \big\| \Big|_{t=0}.$$

(4) $\implies$ (1): This is just the definition of equations (5.10).

(2) $\implies$ (4) : Suppose that $\omega^*(\omega)$ is in the image of J, and fix a $F^0 \colon \tilde{M} \to \mathfrak{g}$ such that (f, F^0) is $\rho_t^{(1)}$-equivariant and $\mathrm{d}F^0 = \omega$. Let $\omega^0 + t\omega_2^0$ be a closed, $\mathrm{Ad}(\rho_t^{(1)})$-valued 1-form representing the cocycle $c + tk$. Then there exists $s \in \mathcal{C}^\infty(\mathcal{V})$ such that $\omega = \omega^0 + \mathrm{d}s$. Then, letting $\omega_2^1 = \omega_2^0 + \mathrm{d}[F^0, s]$, we have that (ω, ω_2^1) is again a closed, $\mathrm{Ad}(\rho_t^{(1)})$-valued 1-form representing $c + tk$. Define:

$$\psi^0 = \omega_2^1 - [F^0, \omega], \quad \psi = \psi^0 + \mathrm{d}\eta,$$

where η is such that $J(\eta) = -\omega^*(\omega) - \mathrm{d}^*\psi^0$. In this way, ψ satisfies (5.10). Finally, define $\omega_2 = \omega_2^1 + \mathrm{d}\eta$. Again, $\omega + t\omega_2$ is a closed $\mathrm{Ad}(\rho_t^{(1)})$-valued representative of $c + tk$. We can apply lemma 3.4.1 to the vector space $V = \mathfrak{g} \otimes \mathbb{R}[t]/(t^2)$, the 1-form $\phi = \omega + t\omega_2$ and the action (5.3). In this way, we construct a pair $F + tF_2 \colon \tilde{M} \to \mathfrak{g} \otimes \mathbb{R}[t]/(t^2)$ such that

$$\mathrm{d}\begin{pmatrix} F \\ F_2 \end{pmatrix} = \begin{pmatrix} \omega \\ \omega_2 \end{pmatrix}; \quad (f, F, F_2) \text{ is } \rho_t^{(2)}\text{-equivariant.}$$

Retracing definitions, we obtain:

$$D_2 \begin{pmatrix} F \\ F_2 \end{pmatrix} = \begin{pmatrix} \omega \\ \omega_2 - [F, \omega] \end{pmatrix} = \begin{pmatrix} \omega \\ \psi - [F - F^0, \omega] \end{pmatrix}.$$

Since $[F - F^0, \omega]$ is harmonic, (F, F_2) is also of harmonic type.

(4) $\iff$ (5): This is lemma 5.4.2. □

Corollary 5.4.7. *If G is a complex algebraic group, we can complete the list by:*

6. *ω minimizes the energy in its orbit $H \times \omega \subseteq \mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$.*

Such a minimum exists if and only if the orbit is closed (that is, ω is a polystable point). Furthermore, every two points ω, ω' in the minimum locus are conjugate by an element of $H \cap K$, and the stabilizer $H' = \mathrm{Stab}_H(\omega)$ is the complexification of $\mathrm{Stab}_{H\cap K}(\omega)$, hence reductive.

Proof. This follows from classic results on moment maps, cfr. e.g. [Kir84], Part 1. In our notations, if we write

$$\mu(\omega)(\xi) = -\frac{i}{2}\int_M \big\langle [\xi, \omega], \omega \big\rangle \quad \xi \in \mathfrak{h}^{\mathfrak{k}},$$

μ is a moment map for the action of the compact subgroup $H \cap K$, and its vanishing is in fact equivalent to $\omega^*(\omega) \perp \mathfrak{h}$, since the vanishing of the above integral for $\xi \in \mathfrak{h}^{\mathfrak{p}}$ (that is, for $i\xi \in \mathfrak{h}^{\mathfrak{k}}$) is always satisfied, as we already remarked that $\omega^*(\omega)$ takes values in $[\mathfrak{p}]$. □

5.5 Obstruction to the existence of w

Thanks to proposition 5.4.6 and corollary 5.2.4, we know that the existence of a solution to (5.10) is a sufficient condition for the existence of a second order harmonic and $\rho_t^{(2)}$-equivariant deformation (v, w) of f. Furthermore, under this hypothesis, every first order deformation v extends, since by theorem 3.6.1 every such v comes from a F, and we have seen that then every F can be extended to a (F, F_2). We now inquire on the converse, namely, whether the existence of w is equivalent to that of F_2. The answer is particularly neat when G is a complex algebraic group. We start with a two lemmas that hold in the real case, as well.

Lemma 5.5.1. *Let $\rho_t^{(2)} = (\rho_0, c, k)$ and $\tilde{\rho}_t^{(2)} = (\rho_0, c, \tilde{k})$ be two second order deformations of ρ_0, coinciding to the first order, and let v be a first order harmonic and $\rho_t^{(1)}$-equivariant deformation of f. Then there exists a $\rho_t^{(2)}$-equivariant and harmonic deformation w if and only if there exists a harmonic and $\tilde{\rho}_t^{(2)}$-equivariant one $\tilde{w}$.*

Proof. The equations for the difference $\tilde{w} - w$ read:

$$\tilde{w}(\gamma\tilde{x}) - w(\gamma\tilde{x}) = \rho_0(\gamma)_*\big(\tilde{w}(\tilde{x}) - w(\tilde{x})\big) + \tilde{k}(\gamma)^{[\mathfrak{p}]} - k(\gamma)^{[\mathfrak{p}]}; \quad \mathcal{J}(\tilde{w} - w) = 0. \tag{5.12}$$

Further, $\tilde{k} - k$ is a 1-cocycle for the adjoint action of ρ_0. Hence, the existence of a section $\tilde{w} - w$ of f^*TN satisfying (5.12) is always assured by theorem 3.6.1, substituting $\tilde{w} - w$ to v and $\tilde{k} - k$ to c. □

Lemma 5.5.2. *Let $\rho_t^{(2)} = (\rho_0, c, k)$ be a second order deformation of ρ_0, and suppose that there exists a $\rho_t^{(2)}$-equivariant and harmonic second order deformation (f, v, w). Then every other $\rho_t^{(1)}$-equivariant and harmonic first order deformation $(f, \tilde{v})$ extends to the second order, as well.*

Proof. Let $F, \tilde{F} \colon \tilde{M} \to N$ be $\rho_t^{(1)}$-equivariant and of harmonic type, such that $\vartheta_{TN}(f, F) = v$ and $\vartheta_{TN}(f, \tilde{F}) = \tilde{v}$. Then there exists a $\xi \in \mathfrak{h}$ such that $\tilde{F} = F + \xi$. Define:

$$\tilde{w} = w + 2\big[F^{[\mathfrak{k}]}, \xi^{\mathfrak{p}}\big] + [\xi^{\mathfrak{k}}, \xi^{\mathfrak{p}}].$$

We claim that $(f, \tilde{v}, \tilde{w})$ is a $\rho_t^{(2)}$-equivariant and harmonic second order deformation of f. For equivariance, just compute:

$$\begin{aligned}\beta_N(\tilde{w})(\gamma\tilde{x}) &= \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(w) + 2\big[c(\gamma)^{[\mathfrak{k}]}, \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v)\big] + \big[c(\gamma)^{[\mathfrak{k}]}, c(\gamma)^{[\mathfrak{p}]}\big]\\ &\quad + k(\gamma)^{[\mathfrak{p}]} + 2\mathrm{Ad}_{\rho_0(\gamma)}\big[F^{[\mathfrak{k}]}, \xi^{\mathfrak{p}}\big] + 2\big[c(\gamma)^{[\mathfrak{k}]}, \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v)\big] + \big[\xi^{\mathfrak{k}}, \xi^{\mathfrak{p}}\big]\\ &= \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(\tilde{w}) + 2\big[c(\gamma)^{[\mathfrak{k}]}, \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(\tilde{v})\big] + \big[c(\gamma)^{[\mathfrak{k}]}, c(\gamma)^{[\mathfrak{p}]}\big]\\ &\quad + k(\gamma)^{[\mathfrak{p}]},\end{aligned}$$

that proves equivariance. For harmonicity, we simply have to remark that, on the one hand, $\mathcal{J}(\tilde{w}) = \mathcal{J}(w)$ and, on the other hand, $R^N(\mathrm{d}f, v)\nabla v = R^N(\mathrm{d}f, \tilde{v})\nabla\tilde{v}$. For the first equality, apply lemma 5.2.2, and conclude using that $[\beta, \xi] = \overset{\text{can}}{\nabla}\xi = 0$ for every $\xi \in \mathfrak{h}$, as we noted in the proof of corollary 1.6.9. The statement of same corollary implies that $\xi^{\mathfrak{p}}$ and $\xi^{\mathfrak{k}}$ are in $\mathfrak{h}$. The second equality is analogous, since after applying β_N to both terms, it becomes

$$\big[[\beta, \beta_N(v)], \overset{\text{can}}{\nabla}\beta_N(v)\big] = \Big[[\beta, \beta_N(v) + \xi^{\mathfrak{p}}], \overset{\text{can}}{\nabla}\big(\beta_N(v) + \xi^{\mathfrak{p}}\big)\Big].$$

□

Corollary 5.5.3. *For extending either an F to a (F, F_2) of harmonic type and $\rho_t^{(2)}$-equivariant, or a v to a (v, w) harmonic and $\rho_t^{(2)}$-equivariant, the obstruction depends on f and c only.*

Definition 5.5.4. We say that a harmonic metric $f \colon \tilde{M} \to N$ is second order-deformable along a first order deformation $\rho_t^{(1)} = (\rho_0, c)$ if there exist a second order deformation $\rho_t^{(2)}$ of $\rho_t^{(1)}$ and a $\rho_t^{(2)}$-equivariant and harmonic deformation (v, w) of f.

In the following, recall that in the complex case G is the complexified of K, that is, $G = \mathbb{K}(\mathbb{C})$ and $\mathbb{G} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{K})$ (scalar restriction à la Weil), so that $G = \mathbb{G}(\mathbb{R})$ and multiplication by i exchanges $[\mathfrak{k}]$ and $[\mathfrak{p}]$. Our aim is to prove that the existence of a $\rho_t^{(2)}$-equivariant (F, F_2) of harmonic type is equivalent to the metric f being deformable along both (ρ_0, c) and (ρ_0, ic). Before proving it, we briefly explain the idea behind this: Ideally, one would like to recover at least $F_2^{[\mathfrak{p}]}$ from w, as one can do in the first order case, where $F^{[\mathfrak{p}]} = \beta_N(v)$. In this case, however, we have

$$F_2^{[\mathfrak{p}]} = \beta_N(w) + \big[F^{[\mathfrak{k}]}, F^{[\mathfrak{p}]}\big],$$

that is, in order to recover $F_2^{[\mathfrak{p}]}$ we need to know w, $F^{[\mathfrak{p}]} = \beta_N(v)$ but also $F^{[\mathfrak{k}]}$. In the complex case, then, since multiplication by i exchanges $[\mathfrak{k}]$ and $[\mathfrak{p}]$, one is naturally led to investigate on $F^{[\mathfrak{k}]} = i\beta_N(\tilde{v})$, where $\tilde{v}$ is a first order (ρ_0, ic)-equivariant harmonic deformation of f (the condition on equivariance is chosen so that $F = F^{[\mathfrak{p}]} + F^{[\mathfrak{k}]}$ becomes $\rho_t^{(1)}$-equivariant). This really works, as the following proposition shows:

Proposition 5.5.5. *Let G be a complex algebraic group, $\rho_t^{(1)} = (\rho_0, c)$ and $\tilde{\rho}_t^{(1)} = (\rho_0, ic)$ be first order deformations of ρ_0 and $f\colon \tilde{M} \to N$ a harmonic metric. Then, for any $\rho_t^{(2)}$ and $\tilde{\rho}_t^{(2)}$ extending themfor e:*

1. *The existence of a $\rho_t^{(2)}$-equivariant (F, F_2) of harmonic type is equivalent to the existence of a $\tilde{\rho}_t^{(2)}$-equivariant $(\tilde{F}, \tilde{F}_2)$ of harmonic type;*
2. *The existence of (F, F_2) as above is equivalent to f being deformable along both $\rho_t^{(1)}$ and $\tilde{\rho}_t^{(1)}$.*

Proof. 1. Since ic is represented by $i\omega$ and $(i\omega)^* = -i\omega^*$, we have $(i\omega)^*(i\omega) = \omega^*(\omega)$, hence proposition 5.4.6, point 2 is invariant under passage from c to ic. Hence the existence of F_2 (point 4 of the same proposition) must be, too. However, in the following we will need an explicit expression for $(\tilde{F}, \tilde{F}_2)$, so we will also give a direct proof of this fact.

Given a (F, F_2) we will construct a $(\tilde{F}, \tilde{F}_2)$. We have already remarked that the existence of such a map depends only on f and $\tilde{c} = ic$, so we are free to choose $\tilde{k}$ as we like. The following is a natural choice:

$$\tilde{\rho}_t^{(2)} = (\rho_0, \tilde{c}, \tilde{k}) = (\rho_0, ic, -k). \tag{5.13}$$

The verification that $(ic, -k)$ is a cocycle for the action (5.3) is immediate. Since we know that (F, F_2) exists, by proposition 5.4.6 we can fix a section

$\eta \in \mathcal{C}^\infty(\mathcal{V})$ such that $J(\eta) = 2\omega^*(\omega)$. Then we claim that $(\tilde{F}, \tilde{F}_2)$ can be defined as follows:

$$(\tilde{F}, \tilde{F}_2) = (iF, -F_2 - \eta).$$

The equations for equivariance and harmonicity of $\tilde{F}$ are evident. For the remaining part, to check equivariance, compute:

$$\begin{aligned}\tilde{F}_2(\gamma\tilde{x}) &= -\mathrm{Ad}_{\rho_0(\gamma)}F_2 - \big[c(\gamma), \mathrm{Ad}_{\rho_0(\gamma)}F\big] - k(\gamma) - \mathrm{Ad}_{\rho_0(\gamma)}(\eta) \\ &= \mathrm{Ad}_{\rho_0(\gamma)}\tilde{F}_2 + \big[ic(\gamma), \mathrm{Ad}_{\rho_0(\gamma)}\tilde{F}\big] - k(\gamma).\end{aligned}$$

To check that $(\tilde{F}, \tilde{F}_2)$ is of harmonic type, recall that we also have to use $\tilde{\omega} = i\omega$ in the flat connection D_2:

$$D_2 \begin{pmatrix} \tilde{F} \\ \tilde{F}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d} & 0 \\ \mathrm{ad}(i\omega) & \mathrm{d} \end{pmatrix} \begin{pmatrix} iF \\ -F_2 - \eta \end{pmatrix} = \begin{pmatrix} i\omega \\ -[\omega, F] - \mathrm{d}F_2 - \mathrm{d}\eta \end{pmatrix}.$$

Now, $\psi = [\omega, F] + \mathrm{d}F_2$ is a 1-form such that $\mathrm{d}^*\psi = -\omega^*(\omega)$, hence, recalling that $(i\omega)^* = -i\omega^*$,

$$D_{2,*}D_2 \begin{pmatrix} \tilde{F} \\ \tilde{F}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}^* & \\ -i\omega^* & \mathrm{d}^* \end{pmatrix} \begin{pmatrix} i\omega \\ -\psi - \mathrm{d}\eta \end{pmatrix} = \begin{pmatrix} i\mathrm{d}^*\omega \\ \omega^*(\omega) + \omega^*(\omega) - \mathrm{d}^*\mathrm{d}\eta \end{pmatrix} = 0.$$

2. The fact that the existence of (F, F_2) implies the existence of both (v, w) and $(\tilde{v}, \tilde{w})$ follows from corollary 5.2.4 and point 1, so we only have to prove the converse. Given a harmonic $\rho_t^{(2)}$-equivariant (v, w) and a harmonic and $\tilde{\rho}_t^{(2)}$-equivariant $(\tilde{v}, \tilde{w})$, where $\tilde{\rho}_t^{(2)}$ is defined as in (5.13), we claim that if we let

$$\eta = \beta_N(w) + 2i\big[\beta_N(\tilde{v}), \beta_N(v)\big] + \beta_N(\tilde{w}),$$

then η is a section of $\mathcal{V}$ such that $J(\eta) = -2\omega^*(\omega)$. This concludes by proposition 5.4.6 and lemma 5.4.5.

To check that η is a section of $\mathcal{V}$, we only have to write down all the equivariances, keeping in mind that multiplication by i exchanges $[\mathfrak{k}]$ and $[\mathfrak{p}]$:

$$\begin{aligned}\beta_N(w)(\gamma\tilde{x}) &= \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(w) + k(\gamma)^{[\mathfrak{p}]} \\ &\quad + 2\big[c(\gamma)^{[\mathfrak{k}]}, \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v)\big] + \big[c(\gamma)^{[\mathfrak{k}]}, c(\gamma)^{[\mathfrak{p}]}\big]; \\ \beta_N(\tilde{w})(\gamma\tilde{x}) &= \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(\tilde{w}) - k(\gamma)^{[\mathfrak{p}]} \\ &\quad + 2\big[ic(\gamma)^{[\mathfrak{p}]}, \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(\tilde{v})\big] + \big[ic(\gamma)^{[\mathfrak{p}]}, ic(\gamma)^{[\mathfrak{k}]}\big]; \\ 2i\big[\beta_N(\tilde{v}), \beta_N(v)\big](\gamma\tilde{x}) &= 2i\mathrm{Ad}_{\rho_0(\gamma)}\big[\beta_N(\tilde{v}), \beta_N(v)\big] + 2i\big[\mathrm{Ad}_{\rho_0(\gamma)}\beta_N(\tilde{v}), c(\gamma)^{[\mathfrak{p}]}\big] \\ &\quad + 2i\big[ic(\gamma)^{[\mathfrak{k}]}, \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v)\big] + 2i\big[ic(\gamma)^{[\mathfrak{k}]}, c(\gamma)^{[\mathfrak{p}]}\big].\end{aligned}$$

Adding these three expressions together, one checks that every term which is not in $\mathrm{Ad}_{\rho_0(\gamma)}(\eta)$ cancels out.

Finally, to compute $J(\eta)$, we preliminarily notice that $F = \beta(v) - i\beta(\tilde{v})$ is $\rho_t^{(1)}$-equivariant and of harmonic type, hence $\mathrm{d}F = \omega$ by lemma 3.4.2. Then, on the one hand

$$J(\eta) = 4\sum_j \left[[\beta(E_j), \beta_N(v)], \overset{\mathrm{can}}{\nabla}_{E_j}\beta_N(v)\right] + \left[[\beta(E_j), \beta_N(\tilde{v})], \overset{\mathrm{can}}{\nabla}_{E_j}\beta_N(\tilde{v})\right]$$
$$+ 4i\big[[\beta(E_j), \beta_N(\tilde{v})], [\beta(E_j), \beta_N(v)]\big] - 4i\big[\overset{\mathrm{can}}{\nabla}_{E_j}\beta_N(\tilde{v}), \overset{\mathrm{can}}{\nabla}_{E_j}\beta_N(v)\big].$$

On the other hand, writing $F^{[\mathfrak{p}]} = \beta_N(v)$ and $F^{[\mathfrak{k}]} = -i\beta_N(\tilde{v})$, one has

$$-\omega^*(\omega) = 2\sum_j \left[\overset{\mathrm{can}}{\nabla}_{E_j}F^{[\mathfrak{k}]} + [\beta(E_j), F^{[\mathfrak{p}]}], \overset{\mathrm{can}}{\nabla}_{E_j}F^{[\mathfrak{p}]} + [\beta(E_j), F^{[\mathfrak{k}]}]\right]$$
$$= 2\sum_j -i\big[\overset{\mathrm{can}}{\nabla}_{E_j}\beta_N(\tilde{v}), \overset{\mathrm{can}}{\nabla}_{E_j}\beta_N(v)\big] - \left[\overset{\mathrm{can}}{\nabla}_{E_j}\beta_N(\tilde{v}), \big[\beta(E_j), \beta_N(\tilde{v})\big]\right]$$
$$+ \left[\big[\beta(E_j), \beta_N(v)\big], \overset{\mathrm{can}}{\nabla}_{E_j}\beta_N(v)\right] - i\left[\big[\beta(E_j), \beta_N(v)\big], \big[\beta(E_j), \beta_N(\tilde{v})\big]\right]$$

□

5.6 Example: an explicit case where w does not exist

We now want to give an explicit example of a first order deformation $\rho_t^{(1)} = (\rho_0, c)$ along which no metric $f\colon \tilde{M} \to N$ is extendable.

Remark 5.6.1. We really need to check every metric f, since, as soon as the dimension fo $H^0(M, \mathrm{Ad}(\rho_t))$ jumps, we expect the existence of a second order deformation to depend on the harmonic metric chosen. For example, suppose that $\rho_t\colon \Gamma \to G$ is a smooth family of representations, and f_t are ρ_t-equivariant and harmonic; suppose further that $H^0(M, \mathrm{Ad}(\rho_t))$ vanishes for every $t \neq 0$, but not for $t = 0$. Then, for every $h \in H$, the map $h \cdot f_0$ is again harmonic and equivariant, but f_t will be the unique harmonic and ρ_t-equivariant map, so $h \cdot f_0$ should be obstructed to a certan order. This obstruction is also reflected in the existence of ψ: If the orbit $H \cdot \omega$ is not a point, the condition of $\mathrm{Ad}_h(\omega)$ being extremal (that is equivalent to $h^{-1} \cdot f$ admitting a ψ) will depend on h.

The setting of our example will be as follows: $\rho_0\colon \Gamma \to \mathrm{SL}(n, \mathbb{R})$ is the trivial representation, and $\rho_t^{(2)} = (\rho_0, c, k)$ is such that $c(\gamma)$ and $k(\gamma)$ are upper strictly upper triangular for every $\gamma \in \Gamma$ (actually, as we noted in lemma 5.5.1, only c is significative). Denote:

$$c(\gamma)_{i,j} = \begin{cases} 0 & \text{if } i \geqslant j \\ \lambda_{i,j}(\gamma) & \text{otherwise;} \end{cases} \qquad k(\gamma)_{i,j} = \begin{cases} 0 & \text{if } i \geqslant j \\ \mu_{i,j}(\gamma) & \text{otherwise.} \end{cases}$$

The harmonic, ρ_0-equivariant maps $f\colon \tilde{M} \to N = G/K = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ all descend to M, hence are constant. So $f(\tilde{x}) = gK$, for some $g \in G$ and $\mathrm{d}f = 0$, hence $\beta = 0$. In particular, the canonical connection is just the flat derivation, which is metric for the positive definite form on $M \times \mathfrak{g}$, which is simply the (twisted) Hilbert–Schimdt scalar product:

$$\big\langle A, B\big\rangle_{gK} = \big\langle \mathrm{Ad}_{g^{-1}}(A), \mathrm{Ad}_{g^{-1}}(B)\big\rangle = \mathrm{trace}\big(A^t \cdot g^t g \cdot B \cdot (g^t g)^{-1}\big).$$

Lemma 5.6.2. *1.* $\langle \mathrm{Ad}_g(A), B\rangle_{eK} = \langle A, \mathrm{Ad}_{g^t}(B)\rangle_{eK}$*;*

2. If d^* *denotes the codifferential with respect to the metric* $\langle\cdot,\cdot\rangle_{eK}$*, then, for every* $g \in G$*,* d^* *commutes with* Ad_g*.*

3. The codifferential d^* *is independent of the metric* $f(\cdot) = gK$*.*

Proof. 1. This is just a computation

$$\begin{aligned}\big\langle \mathrm{Ad}_g(A), B\big\rangle_{eK} &= \mathrm{trace}\Big(\big(gAg^{-1}\big)^t \cdot B\Big) = \mathrm{trace}\Big(A^t g^t B (g^t)^{-1}\Big) \\ &= \big\langle A, \mathrm{Ad}_{g^t}(B)\big\rangle_{eK}\end{aligned}$$

2. Let α be a $\mathfrak{g}$-valued 1-form, $\eta\colon \tilde{M} \to \mathfrak{g}$. Then

$$\big\langle \mathrm{Ad}_g \mathrm{d}^*\alpha, \eta\big\rangle_{eK} = \big\langle \mathrm{d}^*\alpha, \mathrm{Ad}_{g^t}\eta\big\rangle_{eK} = \big\langle \alpha, \mathrm{Ad}_{g^t}\mathrm{d}\eta\big\rangle_{eK} = \big\langle \mathrm{d}^*\mathrm{Ad}_g\alpha, \eta\big\rangle_{eK}.$$

3. Keeping the same notations, denoting by d^{*g} the codifferential with respect to the metric $f(\cdot) \equiv gK$ and using what just proved,

$$\begin{aligned}\big\langle \mathrm{d}^{*g}\alpha, \eta\big\rangle_{gK} &= \big\langle \alpha, \mathrm{d}\eta\big\rangle_{gK} = \big\langle \mathrm{Ad}_{g^{-1}}(\alpha), \mathrm{d}\mathrm{Ad}_{g^{-1}}(\eta)\big\rangle_{eK} \\ &= \big\langle \mathrm{Ad}_{g^{-1}}(\mathrm{d}^*\alpha), \mathrm{Ad}_{g^{-1}}(\eta)\big\rangle_{eK} = \big\langle \mathrm{d}^*\alpha, \eta\big\rangle_{gK}.\end{aligned}$$

□

Corollary 5.6.3. *The Jacobi operator* J *coincide with the Laplace-Beltrami operator for every metric* $f(\cdot) \equiv gK$*.*

Proof. In general, $J(v) = \Delta^{\mathrm{can}}(v) - \mathrm{trace}(R^N(\mathrm{d}f, v)\mathrm{d}f)$, where Δ^{can} is the Laplacian associated to the canonical connection and the metric f. In this case, $\mathrm{d}f = 0$ and $\mathrm{d}^{\mathrm{can}} = \mathrm{d}$, hence $J(v) = \mathrm{d}^{*_g}\mathrm{d}$, in the notations of the lemma. By the last point above, then, $J(v) = \mathrm{d}^*\mathrm{d}$, which is the usual Laplace-Beltrami operator. □

Thanks to this corollary, the definition of harmonic second order deformation (v, w) becomes simply

$$\Delta\beta_N(v) = \Delta\beta_N(w) = 0.$$

Here $\beta_N(v)$ and $\beta_N(w)$ are (symmetric, null-trace) $n \times n$ matrices, and the Laplace-Beltrami operator is taken componentwise.

Lemma 5.6.4. *Let $f_0(\tilde{x}) = eK$ and $f(\tilde{x}) = gK$, for some $g \in G$. By the $G = NAK$ decomposition we can suppose g to be upper triangular, so that $\mathrm{Ad}_g(c(\gamma))$ and $\mathrm{Ad}_g(k(\gamma))$ are still strictly upper triangular. Then setting $\beta_N(v) = \mathrm{Ad}_g(\beta_N(v_0))$ and $\beta_N(w) = \mathrm{Ad}_g(\beta_N(w_0))$ gives an isomorphism of sets*

$$\left\{ (f_0, v_0, w_0) \begin{array}{c} \textit{harmonic and } (\rho_0,c,k) \\ \textit{equivariant second} \\ \textit{order deformations} \end{array} \right\} \leftrightarrow \left\{ (f, v, w) \begin{array}{c} \textit{harmonic and } (\rho_0,\mathrm{Ad}_g(c),\mathrm{Ad}_g(k)) \\ \textit{equivariant second} \\ \textit{order deformations} \end{array} \right\}.$$

In particular, if f_0 is not deformable along (ρ_0, c), no other metric is deformable.

Proof. Corollary 5.6.3 states that the harmonicity condition on the left hand side coincides with that on the right hand side, namely we ask v_0, w_0, v and w to be harmonic functions. The fact that this is preserved by adjunction is point 2. of lemma 5.6.2.

Compatibility of equivariance conditions is essentially trivial, since the projection on $[\mathfrak{k}]$ and $[\mathfrak{p}]$ commute, by definition, with Ad_g (in fact, the non-trivial part is hidden in the NAK-decomposition, that is, the fact that the stated map acts on upper triangular deformations of the trivial representation). For example, denoting $c = \mathrm{Ad}_g(c_0)$ and $k = \mathrm{Ad}_g(k_0)$, the equivariance condition for w is

$$\beta(w(\gamma\tilde{x})) = \beta(w(\tilde{x})) + k(\gamma)^{[\mathfrak{p}]_{gK}} + 2\big[c(\gamma)^{[\mathfrak{k}]_{gK}}, \beta_N(v(\tilde{x}))\big] + \big[c(\gamma)^{[\mathfrak{k}]_{gK}}, c(\gamma)^{[\mathfrak{p}]_{gK}}\big].$$

This is simply the equivariance condition for w_0, after applying Ad_g to every term. □

Thanks to this lemma, from now on we will suppose $f(\tilde{x}) = eK$ for every $\tilde{x}$. In this way, projections to $[\mathfrak{k}]$ and $[\mathfrak{p}]$ become, respectively, anti-symmetrization and symmetrization. Let ω be the harmonic representative of the cohomology class given by c. Then it is a strictly upper triangular matrix, with above-diagonal components $\omega_{i,j} \in \mathcal{H}^1(M, \mathbb{R})$, that is, harmonic 1-forms. Consequently, F is strictly upper triangular, too, and the non-zero components $F_{i,j}$ are harmonic functions. The equivariance relations read $F_{i,j}(\gamma\tilde{x}) = F_{i,j}(\tilde{x}) + \lambda_{i,j}(\gamma)$. Projecting F on $[\mathfrak{p}]$, that is, symmetrizing F, we obtain the expression for $\beta_N(v)$, which is thus a symmetric matrix with zero entries on the diagonal and, for every $i < j$, $\beta_N(v(\tilde{x}))_{i,j} = \beta_N(v(\tilde{x}))_{j,i} = \frac{1}{2}F_{i,j}(\tilde{x})$.

We now want to write down the equations for w to prove that it cannot exist (unless, of course, $c = 0$). We have already noted that harmonicity means simply harmonicity of the components, that is, $\Delta w = 0$. On the other hand, equivariance (definition 5.1.6) reads

$$\beta_N(w(\gamma\tilde{x})) = \beta_N\big(w(\tilde{x})\big) + k(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}} + 2\Big[c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}, \beta_N\big(v(\tilde{x})\big)\Big] + \big[c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}, c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\big].$$

We shall prove that the $(1,1)$ entry of w, which we denote simply by w_1, cannot exist, unless the first line of $c(\gamma)$ is trivial, then we will proceed by induction. Explicitely, for w_1 the equivariance and harmonic equations read:

$$w_1(\gamma\tilde{x}) = w_1(\tilde{x}) + \sum_{j=2}^{n} \lambda_{1,j}(\gamma)F_{1,j}(\tilde{x}) + \frac{1}{2}\sum_{j=2}^{n} \lambda_{1,j}(\gamma)^2; \qquad \Delta(w_1) = 0. \quad (5.14)$$

Lemma 5.6.5. *Let $\tilde{M}$ be a smooth manifold, and Γ a group acting properly discontinuously and cocompactly on $\tilde{M}$. Let $\tau_1, \dots, \tau_m \colon \Gamma \to \mathbb{R}$ be representations, and $F_1, \dots, F_m \colon \tilde{M} \to \mathbb{R}$ harmonic functions such that $F_i(\gamma\tilde{x}) = F_i(\tilde{x}) + \tau_i(\gamma)$. If there exists a harmonic function $h \colon \tilde{M} \to \mathbb{R}$ such that*

$$h(\gamma\tilde{x}) = h(\tilde{x}) + \sum_{i=1}^{m} \tau_i(\gamma)F_i(\tilde{x}) + \frac{1}{2}\sum_{i=1}^{m} \tau_i(\gamma)^2, \qquad (5.15)$$

all of the F_i's must be constant, hence all of the τ_i's must be trivial.

Proof. One has only to remark that any h as in (5.15) has the same kind of equivariance as $\frac{1}{2}\sum_{i=1}^{m} F_i(\cdot)^2$. Indeed:

$$\frac{1}{2}F_i(\gamma\tilde{x})^2 = \frac{1}{2}\Big(F_i(\tilde{x})^2 + 2\tau_i(\gamma)F_1(\tilde{x}) + \tau_i(\gamma)^2\Big).$$

In particular, $\sum_{i=1}^{m} \frac{1}{2}F_i(\cdot)^2 - h$ is Γ-invariant, that is, it is defined on the compact manifold $M = \tilde{M}/\Gamma$. This function is the difference of a subharmonic function and a harmonic one, hence it is subharmonic; thus, by the maximum principle it must be constant. But then $\sum_{i=1}^{m} F_i(\cdot)^2$ would be harmonic as well, hence every F_i must be constant (the Laplacian of this function being the sum of the squared norms of the gradients of F_i). □

Thanks to this lemma, $\lambda_{1,j}(\gamma)$ must vanish for every j; then one can proceed by induction, since, under this vanishing hypothesis, the equivariance equation for the $(2,2)$ entry of w, which we denote by w_2, becomes:

$$w_2(\gamma\tilde{x}) = w_2(\tilde{x}) + \sum_{j=3}^{n} \lambda_{2,j}(\gamma) F_{2,j}(\tilde{x}) + \frac{1}{2}\sum_{j=3}^{n} \lambda_{2,j}(\gamma)^2.$$

Proceeding as above, we find by recurrence that $c(\gamma)$ must vanish. In this case, of course, the second order deformation exists, since the equations for w are formally the same as those for v (exchanging the $\mu_{i,j}$ to the $\lambda_{i,j}$).

Example 5.6.6. When $M = S^1$ and $G = \mathrm{SL}(2,\mathbb{R})$, then the family of representations takes the form

$$\rho_t(1) = g_t = \begin{pmatrix} 1 & \lambda t \\ 0 & 1 \end{pmatrix}.$$

Then the "natural" candidates for ρ_t-equivariant smooth maps are the curves $f_t(\tilde{x}) = 1 + t\lambda x$, which are indeed such that $f_t(\tilde{x}+1) = g_t \cdot f_t(\tilde{x}) = f_t(\tilde{x}) + t\lambda$. Clearly, these are not geodesics for $t \neq 0$, but they are (rescaled) horocycles, as $\|\nabla_{\dot{f}_t}\dot{f}_t\| = t^2\lambda^2$.

5.7 Conclusions

We summarize here the main results of the section.

Theorem 5.7.1. *Let $\rho_t^{(2)} = (\rho_0, c, k)$ be a second order deformation of ρ_0, and f a harmonic metric. If one of the equivalent conditions in proposition 5.4.6 holds, then the map*

$$\vartheta_{J^2N} \colon \left\{ \begin{pmatrix} F \\ F_2 \end{pmatrix} \begin{array}{l} \rho_t^{(2)}\text{-equivariant} \\ \text{of harmonic type} \end{array} \right\} \longrightarrow \left\{ \begin{pmatrix} v \\ w \end{pmatrix} \begin{array}{l} \rho_t^{(2)}\text{-equivariant} \\ \text{and harmonic} \end{array} \right\}$$

$$\begin{pmatrix} F \\ F_2 \end{pmatrix} \longmapsto \Big(\vartheta_{TN}(f,F), \vartheta_{TN}\big(f, F_2 + [F^{[\mathfrak{k}]}, F^{[\mathfrak{p}]}]\big)\Big)$$

is surjective, and in fact every $\rho_t^{(1)}$-equivariant and harmonic first order deformation (f,v) extends to a second order $\rho_t^{(2)}$-equivariant and harmonic (f,v,w).

When G is a complex algebraic group, the condition above is in fact equivalent to the existence of two harmonic and equivariant second order deformations, one for (ρ_0, c) and the other for (ρ_0, ic). In this case, up to changing f to $h^{-1}f$, for some $h \in H$, the condition can be satisfied if and only if the orbit $H \cdot \omega$ is closed in $\mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$.

Proof. The only statement that does not appear elsewhere in the section is the surjectivity, but this follows easily from point 5 of propositon 5.4.6. Let (v, w) be any $\rho_t^{(2)}$-equivariant and harmonic second order deformation of f. Then $v = \vartheta_{TN}(F)$ for some F, which by hypothesis can be extended to a $\rho_t^{(2)}$-equivariant $(F, \tilde{F}_2)$ of harmonic type. Projecting we obtain $(v, \tilde{w}) = \vartheta_{J^2N}(F, \tilde{F}_2)$, another second order harmonic and $\rho_t^{(2)}$-equivariant deformation. We claim that

$$(F, F_2) = \big(F, \tilde{F}_2 + \beta_N(w - \tilde{w})\big) \mapsto \vartheta_{J^2N}(F, F_2) = (v, w).$$

Indeed, since w and $\tilde{w}$ extend the same v, defining $\xi = \beta_N(w - \tilde{w})$ we have $J(\xi) = 0$ and $\gamma^*\xi = \mathrm{Ad}_{\rho_0(\gamma)}(\xi)$, hence $\xi \in \mathfrak{h}$. Adding to $(F, \tilde{F}_2)$ such an element $(0, \xi)$ gives another $\rho_t^{(2)}$-equivariant map of harmonic type, and the result follows easily. □

Let us now summarize all of the main results about existence of second order harmonic and equivariant deformations of a harmonic metric $f\colon \tilde{M} \to N$, in the case of a complex group G.

Proposition 5.7.2. *Let G be a complex algebraic group and $\rho_t^{(2)} = (\rho_0, c, k)\colon \Gamma \to J^2G$ a second order deformation of a representation ρ_0. Then, in the following list, the same numbers with different priming denote equivalent statements, and numbers are decreasing in strength, i.e., (1) $\implies$ (2) $\implies$ (3).*

(1) The $\mathbb{R}[t]/(t^2)$-module $H^0(M, \mathrm{Ad}(\rho_t^{(1)}))$ is flat.

(1′) $H = H'$, that is, ω is a fixed point of the action $H \circlearrowright \mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$, or equivalently we have an exact sequence

$$0 \to \mathfrak{h} \xrightarrow{\times t} H^0(M, \mathrm{Ad}(\rho_t^{(1)})) \xrightarrow{\bmod t} \mathfrak{h} \to 0.$$

(1″) We have the inclusion $\mathfrak{h}^{\mathfrak{p}} \subset \mathfrak{h}'$.

(1‴) Every harmonic ρ_0-equivariant map $f\colon \tilde{M} \to N$ is deformable along both (ρ_0, c) and (ρ_0, ic).

(1⁗) For every harmonic ρ_0-equivariant map $f\colon \tilde{M} \to N$, for any two second order deformations $\rho_t^{(2)}$, $\tilde{\rho}_t^{(2)}$ of $\rho_t^{(1)} = (\rho_0, c)$ and $\tilde{\rho}_t^{(1)} = (\rho_0, ic)$, respectively, and for any two $v, \tilde{v}$, first order harmonic $\rho_t^{(1)}$-equivariant (resp. $\tilde{\rho}_t^{(1)}$-equivariant) deformations for f, there exist w and $\tilde{w}$, second order harmonic and $\rho_t^{(2)}$-equivariant (resp. $\tilde{\rho}_t^{(2)}$-equivariant) deformations of (f, v) (resp. $(f, \tilde{v})$).

(2) The orbit $H \cdot \omega$ is closed.

(2′) There exists a harmonic ρ_0-equivariant map $f\colon \tilde{M} \to N$ which is deformable along both (ρ_0, c) and (ρ_0, ic).

(2″) There exists a harmonic ρ_0-equivariant map $f\colon \tilde{M} \to N$ such that for any two second order deformations $\rho_t^{(2)}$, $\tilde{\rho}_t^{(2)}$ of $\rho_t^{(1)} = (\rho_0, c)$ and $\tilde{\rho}_t^{(1)} = (\rho_0, ic)$, respectively, and for any two $v, \tilde{v}$, first order harmonic $\rho_t^{(1)}$-equivariant (resp. $\tilde{\rho}_t^{(1)}$-equivariant) deformations for f, there exist w and $\tilde{w}$, second order harmonic and $\rho_t^{(2)}$-equivariant (resp. $\tilde{\rho}_t^{(2)}$-equivariant) deformations of (f, v) (resp. $(f, \tilde{v})$).

(3) The stabilizing subgroup H' is reductive.

Proof. For example, $(1') \implies (2)$ is trivial, and $(2) \implies (3)$ is corollary 5.4.7. Lemma 5.3.3 gives the equivalence in $(1')$.

$(1) \iff (1')$: Since the only non-trivial ideal of $A = \mathbb{R}[t]/(t^2)$ is (t), letting $\mathcal{M}$ denote the A-module $H^0(M, \mathrm{Ad}(\rho_t^{(1)}))$, flatness of $\mathcal{M}$ means injectivity of $(t) \otimes_A \mathcal{M} \to \mathcal{M}$. This map sends to zero the elements of the form $t \otimes t\eta$, where $t\eta \in \mathcal{M}$, so flatness is equivalent to asking that

$$\forall t\eta \in \mathcal{M}, \quad t \otimes t\eta = 0 \in (t) \otimes_A \mathcal{M}.$$

On the one hand, $t \otimes t\eta = 0$ in $(t) \otimes \mathcal{M}$ is equivalent to $\eta \in \mathcal{M}$; on the other hand, from (5.9) we see that $t\eta \in \mathcal{M} \iff \eta \in \mathfrak{h}$, so that flatness is in turn equivalent to $\mathfrak{h} \subseteq \mathcal{M}$. This last property is exactly the surjectivity of the map $H^0(M, \mathrm{Ad}(\rho_t^{(1)})) \to \mathfrak{h}$, that is, the exactness of the sequence of point $(1')$.

$(1') \iff (1'')$: One direction is trivial; the other one follows by $\mathfrak{h}$ and $\mathfrak{h}'$ being complex Lie algebras, so that $\mathfrak{h} = i\mathfrak{h}$ and $\mathfrak{h}' = i\mathfrak{h}'$. On the other hand, multiplication by i exchanges $[\mathfrak{p}]$ and $[\mathfrak{k}]$, so that $i\mathfrak{h}^{\mathfrak{p}} = \mathfrak{h}^{\mathfrak{k}}$. Hence from $\mathfrak{h}^{\mathfrak{p}} \subseteq \mathfrak{h}'$ it follows also $\mathfrak{h}^{\mathfrak{k}} \subseteq \mathfrak{h}'$, that is, $\mathfrak{h} \subseteq \mathfrak{h}'$.

$(1') \implies (1''')$: The equality $\mathfrak{h} = \mathfrak{h}'$ means that for every $\xi \in \mathfrak{h}$, the harmonic 1-form $[\omega, \xi]$ vanishes. In particular, $\langle \omega, [\omega, \xi] \rangle = 0$, that is, $\omega^*(\omega) \perp \mathfrak{h}$. By proposition 5.4.6 we can construct a $\rho_t^{(2)}$-equivariant (F, F_2) of harmonic type, hence, by proposition 5.5.5, every harmonic matric is deformable along both $\rho_t^{(1)}$ and $\tilde{\rho}_t^{(1)}$.

$(1''') \iff (1'''')$: This follows at once from lemmas 5.5.1 and 5.5.2.

$(1''') \implies (1')$: By proposition 5.5.5, we know that a (F, F_2) exists for every harmonic metric f, that is, the equivalent statements of proposition 5.4.6 hold. Fix one such metric f_0; then every other is obtained as $f = h^{-1} f_0$, for $h \in H$. We thus have that ω is a critical point of $\int \|\omega\|_f^2$ in its H-orbit for every such f, that implies that $\int \|\mathrm{Ad}_h\omega\|^2_{f_0}$ is independent of h. That is, the L^2-norm is constant on the orbit; by corollary 5.4.7, the minimum locus is a $(H \cap K)$-orbit, hence $H = (H \cap K) \cdot H'$. But this implies $H = H'$, since, again by corollary 5.4.7, H' is reductive, hence its Lie algebra decomposes as $\mathfrak{h}' = \mathfrak{h}^{\mathfrak{k}'} \oplus \mathfrak{h}^{\mathfrak{p}'}$. Suppose, by contradiction, we had a $\xi \in \mathfrak{h}^{\mathfrak{k}} \backslash \mathfrak{h}'$, and write $i\xi = \eta^{\mathfrak{k}} + \eta'$, for some $\eta^{\mathfrak{k}} \in \mathfrak{h}^{\mathfrak{k}}$ and $\eta' \in \mathfrak{h}'$. Since $\mathfrak{h}^{\mathfrak{k}} \oplus \mathfrak{h}^{\mathfrak{p}}$ is a direct sum decomposition, ξ must be the projection of η' to $\mathfrak{h}^{\mathfrak{p}'}$, hence it must belong to $\mathfrak{h}'$. But $\mathfrak{h}'$ being complex, this implies also $\xi \in \mathfrak{h}'$.

(2) $\iff$ (2′) $\iff$ (2″): One simply needs to replicate the same arguments done for the equivalences of (1′), (1″) and (1‴), after having chosen the metric for which ω is a minimum of the energy. Such a metric exists since $H \cdot \omega$ is closed, thanks to corollary 5.4.7. □

In the real case, the picture is slightly less satisfying. Conditions (1), (1′) are still equivalent, but, in general, stronger than (1″). Any of these conditions implies the non-obstruction of (ρ_0, c), but, of course, we cannot speak about (ρ_0, ic). Furthermore, to my knowledge the theorem relating closedness of the orbit, existence of minima of the norm and reductivity of the stabilizer is only valid in the complex setting.

Both in the complex and in the real case, it is currently unknown wheter there exists any "exceptional" deformation (v, w) which is not induced by a $\rho_t^{(2)}$-equivariant (F, F_2) of harmonic type; in the complex case, of course, this is equivalent to the existence of a metric f which is deformable along $\rho_t^{(1)} = (\rho_0, c)$ but not along (ρ_0, ic).

Example 5.7.3. Working in $\mathrm{SL}(n, \mathbb{C})$ instead of $\mathrm{SL}(n, \mathbb{R})$, all of the examples given by "unipotent deformations of the identity" in section 5.6 are made of metrics which are not deformable neither along (ρ_0, c) nor along (ρ_0, ic). This is clear, since the proof is independent of the specific c chosen among the strictly upper triangular ones; more explicitly, one can see that in this case the equation (5.14) remains unchanged after passing from c to ic and from F to iF (the minus sign appearing is taken care of by the switching of $[\mathfrak{k}]$ and $[\mathfrak{p}]$, which, for upper triangular matrices, is exactly a change a sign).

5.8 Motivation for the definitions

First of all, we motivate the definitions for second order harmonic and $\rho_t^{(2)}$-equivariant deformation, proving that if we are given a family of representations $\rho_t \colon \tilde{M} \to G$ and $f_t \colon \tilde{M} \to N$ is a smooth family of harmonic (resp.

ρ_t-equivariant) maps for $t \in (-\varepsilon, \varepsilon)$, then defining

$$v = \frac{\partial f_t}{\partial t}\Big|_{t=0}; \quad w = \frac{D}{\partial t}\frac{\partial f_t}{\partial t}\Big|_{t=0}$$

gives a harmonic (resp. $\rho_t^{(2)}$-equivariant) second order deformation of f.

Proposition 5.8.1. *Let $\mathcal{F}\colon \tilde{M} \times (-\varepsilon, \varepsilon) \to N$, be any smooth one-parameter family of smooth maps, denote $f_t(\tilde{x}) = \mathcal{F}(\tilde{x}, t)$ and let $v = \frac{\partial f_t}{\partial t}\big|_{t=0}$, $w = \frac{D}{\partial t}\frac{\partial f_t}{\partial t}\big|_{t=0}$. Suppose further that f_0 is harmonic. Then, in terms of a local frame $\{\frac{\partial}{\partial x_j}\}$,*

$$\frac{D}{\partial t}\frac{D}{\partial t}\tau_{f_t}\Big|_{t=0} = -\mathcal{J}(w) + 4\sum_{j,k} g^{jk}\Big(R^N(\frac{\partial f}{\partial x_j}, v)\overset{N}{\nabla}_{\frac{\partial}{\partial x_k}}(v)\Big).$$

Proof. This is just computation in the spirit of proposition 3.3.2. Starting from the result of that same proposition (or rather from step (III) of its proof), we may write, in terms of a local orthonormal frame $\{E_s\}$,

$$\frac{D}{\partial t}\tau(f_t) = \sum_s \overset{N}{\nabla}_{E_s}\overset{N}{\nabla}_{E_s}\frac{\partial f_t}{\partial t} + R^N\Big(\mathrm{d}f_t(E_s), \frac{\partial f_t}{\partial t}\Big)\mathrm{d}f_t(E_s).$$

We derive again this expression with respect to t, making use twice of the formula for the curvature:

$$\begin{aligned}\frac{D}{\partial t}\overset{N}{\nabla}_{E_s}\overset{N}{\nabla}_{E_s}\frac{\partial f_t}{\partial t} &= \overset{N}{\nabla}_{E_s}\overset{N}{\nabla}_{E_s}\frac{D}{\partial t}\frac{\partial f_t}{\partial t} + \overset{N}{\nabla}_{E_s}R^N(\mathrm{d}f(E_s), \frac{\partial f_t}{\partial t})\frac{\partial f_t}{\partial t} \\ &\quad + R^N(\mathrm{d}f(E_s), \frac{\partial f_t}{\partial t})\overset{N}{\nabla}_{E_s}\frac{\partial f_t}{\partial t}.\end{aligned}$$

Now N is (locally) symmetric, hence $\overset{N}{\nabla}(R^N) = 0$. We can thus distribute the covariant derivative to every term of the curvature, obtaining:

$$\begin{aligned}\frac{D}{\partial t}\overset{N}{\nabla}_{E_s}\overset{N}{\nabla}_{E_s}\frac{\partial f_t}{\partial t}\Big|_{t=0} &= \overset{N}{\nabla}_{E_s}\overset{N}{\nabla}_{E_s}w + R^N(\mathrm{d}f(E_s), v)\overset{N}{\nabla}_{E_s}v + R^N(\tau(f), v)v \\ &\quad + R^N(\mathrm{d}f(E_s), \overset{N}{\nabla}_{E_s}v)v + R^N(\mathrm{d}f(E_s), v)\overset{N}{\nabla}_{E_s}v.\end{aligned}$$

Again since R^N is parallel, we get

$$\begin{aligned}\frac{D}{\partial t}\Big(R^N\big(\mathrm{d}f_t(E_s), \frac{\partial f_t}{\partial t}\big)\mathrm{d}f_t(E_s)\Big)\Big|_{t=0} &= R^N(\overset{N}{\nabla}_{E_s}v, v)\mathrm{d}f(E_s) \\ &\quad + R^N(\mathrm{d}f(E_s), w)\mathrm{d}f(E_s) + R^N(\mathrm{d}f(E_s), v)\overset{N}{\nabla}_{E_s}v.\end{aligned}$$

Summing up these two expressions and using that $\tau(f) = 0$, we have

$$\frac{D}{\partial t}\frac{D}{\partial t}\tau_{f_t}\Big|_{t=0} = -\mathcal{J}(w) + \sum_s 3R^N(\mathrm{d}f(E_s), v)\overset{N}{\nabla}_{E_s} v + R^N(\mathrm{d}f(E_s), \overset{N}{\nabla}_{E_s} v)v$$
$$+ R^N(\overset{N}{\nabla}_{E_s} v, v)\mathrm{d}f(E_s).$$

Recall the symmetry of the Riemannian tensor:

$$R(a, b)c + R(b, c)a + R(c, a)b = 0;$$

applying this to $a = \mathrm{d}f(E_s)$, $b = \overset{\mathrm{can}}{\nabla}_{E_s}$ and $c = v$, the sum of the last two terms coincides with $-R^N(v, \mathrm{d}f(E_s))\overset{N}{\nabla}_{E_s} v = R^N(\mathrm{d}f(E_s), v)\overset{N}{\nabla}_{E_s} v$. □

Lemma 5.8.2. *Let $\mathcal{F}\colon \tilde{M} \times (-\varepsilon, \varepsilon) \to N$ be a smooth family of smooth maps, such that $f_t = \mathcal{F}(\cdot, t)$ is ρ_t-equivariant. Define v, w as above; then the second order deformation (f, v, w) is $\rho_t^{(2)}$-equivariant, according to definition 5.1.6.*

Proof. We have already proved the statement regarding f and v, so it only remains to prove the formula for $w(\gamma\tilde{x})$. It will be somehow more convenient to work with $\beta_N(w)$, that is, to prove that

$$\beta_N(w)(\gamma\tilde{x}) = \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(w) + k(\gamma)^{[\mathfrak{p}]} + 2\big[c(\gamma)^{[\mathfrak{k}]}, \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v)\big] + \big[c(\gamma)^{[\mathfrak{k}]}, c(\gamma)^{[\mathfrak{p}]}\big]. \tag{5.16}$$

We will denote by $\frac{\partial}{\partial t}$ the derivation with respect to the flat connection and $\frac{D}{\partial t}$ the covariant derivation with respect to the canonical connection, so that, for every vector field $X_t = X_t(\tilde{x})$ on $\tilde{M} \times (-\varepsilon, \varepsilon)$,

$$\frac{D}{\partial t}X_t\Big|_{t=0} = \frac{\partial X_t}{\partial t}\Big|_{t=0} - \Big[f_t^*\beta_N\big(\frac{\partial}{\partial t}\big), X_t\Big]\Big|_{t=0} = \frac{\partial X_t}{\partial t}\Big|_{t=0} - \Big[\beta_N(v), X_0\Big]. \tag{5.17}$$

First notice that

$$\beta_N\Big(\frac{D}{\partial t}\frac{\partial f_t}{\partial t}\Big)\Big|_{t=0} = \frac{D}{\partial t}\beta_N\big(\frac{\partial f_t}{\partial t}\big)\Big|_{t=0} = \frac{\partial}{\partial t}\Big(\beta_N\big(\frac{\partial f_t}{\partial t}\big)\Big)\Big|_{t=0},$$

since in this case the second term of (5.17) vanishes. Hence, using the equivariance condition on v,

$$\beta_N(w)(\gamma\tilde{x}) = \frac{\partial}{\partial t}\Big(\beta_N\big(\frac{\partial f_t}{\partial t}\big)\Big)(\gamma\tilde{x})\Big|_{t=0} = \frac{\partial}{\partial t}\Big(\mathrm{Ad}_{\rho_t(\gamma)}\beta_N\big(\frac{\partial f_t}{\partial t}\big) + c_t(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\Big)\Big|_{t=0}.$$

Then, on the one hand

$$\frac{\partial}{\partial t}\mathrm{Ad}_{\rho_t(\gamma)}\beta_N\big(\frac{\partial f_t}{\partial t}\big)\Big|_{t=0} = \mathrm{Ad}_{\rho_0(\gamma)}\beta_N(w) + \big[c(\gamma), \mathrm{Ad}_{\rho_0(\gamma)}(\beta_N(v))\big]; \tag{5.18}$$

on the other hand, to compute the derivative of $c_t(\gamma)^{[\mathfrak{p}]}$ we switch back to the covariant derivative by means of (5.17), and use that the canonical connection commutes with the projection to $[\mathfrak{p}]$. Thus we get:

$$\frac{\partial}{\partial t}\Big(c_t(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\Big) = \Big(\frac{D}{\partial t}c_t(\gamma)\Big)^{[\mathfrak{p}]}_{\gamma\tilde{x}} + \Big[\beta_N(v)_{\gamma\tilde{x}}, c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\Big].$$

Using the equivariance for v, the second term becomes

$$\Big[\beta_N(v)_{\gamma\tilde{x}}, c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\Big] = \Big[\mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v)_{\tilde{x}} + c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}, c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}\Big], \tag{5.19}$$

which reduces to $[\mathrm{Ad}_{\rho_0(\gamma)}\beta_N(v)_{\tilde{x}}, c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}]$. The first one becomes

$$\begin{aligned}\Big(\frac{D}{\partial t}c_t(\gamma)\Big)^{[\mathfrak{p}]}_{\gamma\tilde{x}} &= \Big(\frac{\partial c_t(\gamma)}{\partial t}\Big|_{t=0} - [\beta(v)_{\gamma\tilde{x}}, c(\gamma)]\Big)^{[\mathfrak{p}]}_{\gamma\tilde{x}} \\ &= k(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}} - \big[\beta(v)_{\gamma\tilde{x}}, c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}\big] \\ &= k(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}} - \big[\mathrm{Ad}_{\rho_0(\gamma)}\beta(v)_{\tilde{x}}, c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}\big] - \big[c(\gamma)^{[\mathfrak{p}]}_{\gamma\tilde{x}}, c(\gamma)^{[\mathfrak{k}]}_{\gamma\tilde{x}}\big].\end{aligned} \tag{5.20}$$

Adding together (5.18), (5.19) and (5.20), the terms of the type $[c(\gamma)^{[\mathfrak{p}]}, \beta_N(v)]$ cancel out, and those of type $[c(\gamma)^{[\mathfrak{k}]}, \beta_N(v)]$ sum together, proving (5.16). $\square$

Finally, as a motivation for equations (5.10), hence for the operators D_2 and $D_{2,*}$, we analyze the specially simple case when, defining for every $t \in (-\varepsilon, \varepsilon)$ and the 1-form ω_t as the harmonic representative of $c_t(\gamma) = \frac{\partial \rho_t(\gamma)}{\partial t} \cdot \rho_t(\gamma)^{-1}$, then ω_t depends smoothly on t. We adopt the notational convention that, if $\alpha \in \mathcal{A}^1(\mathcal{V})$ is a $\mathcal{V}$-valued form, then $\tilde{\alpha} = \tilde{\pi}^*\alpha$ is its pull-back to $\tilde{M}$, which is thus $\mathfrak{g}$-valued. Define:

$$\tilde{\omega}_2 = \frac{\partial\tilde{\omega}_t}{\partial t}\Big|_{t=0}; \quad \tilde{\psi} = \tilde{\omega}_2 - [F, \tilde{\omega}].$$

Then $\tilde{\psi}$ is equivariant, thus descends to a $\psi \in \mathcal{A}^1(\mathcal{V})$. To see this, one derives $\gamma^*\tilde{\omega}_t = \mathrm{Ad}_{\rho_t(\gamma)}\tilde{\omega}_t$ to obtain, thanks to equivariance of F,

$$\gamma^*\tilde{\psi} = \mathrm{Ad}_{\rho(\gamma)}(\tilde{\omega}_2) + \big[c(\gamma), \mathrm{Ad}_{\rho(\gamma)}\tilde{\omega}\big] - \big[\mathrm{Ad}_{\rho(\gamma)}F + c(\gamma), \mathrm{Ad}_{\rho(\gamma)}\tilde{\omega}\big] = \mathrm{Ad}_{\rho(\gamma)}\tilde{\psi}.$$

We claim that this 1-form ψ satisfies equations (5.10). The first of those equations follows from $\mathrm{d}\tilde{\omega}_2 = 0$, $\mathrm{d}F = \omega$ and $\mathrm{d}\omega = 0$. To prove the second one, derive with respect to t the equality $0 = \int_M \langle \omega_t, \mathrm{d}\xi_t\rangle = \int_\Sigma \langle \tilde{\omega}_t, \mathrm{d}\tilde{\xi}_t\rangle$, where Σ is a fundamental domain for the action of Γ on $\tilde{M}$ and ξ_t is a section of $\mathcal{V}_t$ (the flat bundle associated to the local system $\mathrm{Ad}(\rho_t)$), to obtain

$$\begin{aligned}0 = \int_\Sigma \frac{\partial}{\partial t}\langle \tilde{\omega}_t, \mathrm{d}\tilde{\xi}_t\rangle &= \int_\Sigma \Big\langle \frac{D}{\partial t}\tilde{\omega}_t\Big|_{t=0}, \mathrm{d}\tilde{\xi}\Big\rangle + \Big\langle \tilde{\omega}, \frac{D}{\partial t}\mathrm{d}\tilde{\xi}_t\Big|_{t=0}\Big\rangle \\ &= \int_\Sigma \big\langle \tilde{\psi} + [F^{[\mathfrak{k}]}, \tilde{\omega}], \mathrm{d}\tilde{\xi}\big\rangle + \Big\langle \tilde{\omega}, \frac{\partial}{\partial t}\mathrm{d}\tilde{\xi}_t\Big|_{t=0} - [F^{[\mathfrak{p}]}, \mathrm{d}\tilde{\xi}]\Big\rangle,\end{aligned}$$

where we have used that $\frac{D}{\partial t}\big|_{t=0} = \frac{\partial}{\partial t}\big|_{t=0} - [f_t^*\beta_N(\frac{\partial}{\partial t})\big|_{t=0}, \cdot] = \frac{\partial}{\partial t}\big|_{t=0} - [F^{[\mathfrak{p}]}, \cdot]$ and the definition of ψ. Now d and $\frac{\partial}{\partial t}$ commute and the Lie bracket by $F^{[\mathfrak{k}]}$ is anti-selfadjoint, so that

$$0 = \int_\Sigma \langle \mathrm{d}^*\tilde{\psi}, \tilde{\xi}\rangle + \langle \tilde{\omega}, \mathrm{d}\frac{\partial \tilde{\xi}_t}{\partial t}\big|_{t=0} - [F, \mathrm{d}\tilde{\xi}]\rangle.$$

Finally, since $\gamma^*\tilde{\xi}_t = \mathrm{Ad}_{\rho_t(\gamma)}(\tilde{\xi}_t)$, the quantity $\frac{\partial \tilde{\xi}_t}{\partial t}\big|_{t=0} - [F, \tilde{\xi}]$ is the pullback of a section of $\mathcal{V}$, so that $\mathrm{d}^*\omega = 0$ implies $\int_\Sigma \big\langle \tilde{\omega}, \mathrm{d}(\frac{\partial \tilde{\xi}_t}{\partial t}\big|_{t=0} - [F, \tilde{\xi}])\big\rangle = 0$. Thus, recalling that $\mathrm{d}F = \omega$, the expression simplifies to give

$$0 = \int_M \langle \mathrm{d}^*\psi, \xi\rangle + \langle \omega, [\omega, \xi]\rangle = \int_M \langle \mathrm{d}^*\psi, \xi\rangle - \langle \omega^*(\omega), \xi\rangle.$$

In this case, of course, F_2 exists, and it is given by $\frac{\partial F_t}{\partial t}\big|_{t=0}$, where $\mathrm{d}F_t = \omega_t$ and $F_t(\gamma\tilde{x}) = \mathrm{Ad}_{\rho_t(\gamma)}F_t(\tilde{x}) + c_t(\gamma)$.

Chapter 6

Second variation of the energy and minimality of VHS

Introduction au chapitre

Dans ce chapitre on va appliquer les résultats du chapitre 5 à l'étude de la variation seconde de l'énergie. Pour pouvoir appliquer la théorie développée, on va supposer dans la suite que l'application harmonique ρ_0-équivariante f soit déformable le long d'une déformation au second ordre $\rho_t^{(2)}$ au sens fort, c'est-à-dire, que les conditions équivalentes de la Proposition 5.4.6 soient vérifiées. On peut alors parler de la 1-forme ψ issue de (F, F_2); en termes de cette 1-forme, on donne une formule pour la variation seconde de l'énergie ressemblante à celle pour le premier ordre. En effet, on obtient que cette variation est égale à $\int \langle \psi, \beta \rangle + \|\omega^{[\mathfrak{p}]}\|^2$, l'indice $[\mathfrak{p}]$ indiquant la partie autoadjointe de ω. Comme on avait fait pour l'énergie et pour sa variation première, on donne, dans le cas Kähler, une interprétation cohomologique de cette formule, qui implique en particulier qu'elle ne dépend pas de la forme de Kähler choisie dans sa classe de cohomologie.

La première application de la formule de la variation seconde concerne la plurisousharmonicité de l'énergie. Ici, G doit être un groupe complexe, mais à M on ne demande que d'être riemannienne, et l'hypothèse de déformabilité "forte" est en fait automatique, grâce à la Proposition 5.5.5. On prouve alors que l'énergie est plurisousharmonique par rapport à la structure complexe de $\mathbb{M}_{\mathrm{B}}(M, G)$.

Deuxièmement, on s'intéresse aux valeurs propres de la matrice hessienne de l'énergie dans les points critiques. Ici on suppose que $M = X$ soit Kähler pour pouvoir appliquer la théorie des variations de structures de Hodge. Le résultat est une généralisation de celui démontré par Hitchin en [Hit92] et on va adopter une notation similaire à la sienne pour mieux décrire le théorème. Posons alors $\dot{A} = (\omega^{[\mathfrak{k}]})''$ (la partie de tipe $(0,1)$ de la partie anti-autoadjointe de ω) et $\dot{\Phi} = (\omega^{[\mathfrak{p}]})'$. Écrivons $\dot{A} = \sum \dot{A}^{-p,p}$ et $\dot{\Phi} = \sum \dot{\Phi}^{-p,p}$ en correspondance à la variation de structure de Hodge de poids 0 induite sur $\tilde{X} \times_\Gamma \mathfrak{g}$, d'une façon à ce que le générateur infinitesimal γ de l'action de S^1 agisse avec un poids de ip sur $\dot{A}^{-p,p}$ et sur $\dot{\Phi}^{-p,p}$. On démontre alors que la matrice hessienne de l'énergie a pour valeurs propres $-2p$ dans la direction $\dot{A}^{p,p}$ et $2(1-p)$ dans la direction $\dot{\Phi}^{-p,p}$ (le facteur 2 ajouté par rapport à Hitchin est dû à un choix différent dans la métrique). Cette formule s'explique plus clairement dans le cas où l'on suppose que la déformation se produise dans le sous-groupe $G_0 < G$ donné par la monodromie de ρ_0; on a alors une décomposition de l'énergie par rapport à la structure de Hodge-Deligne de poids 1 sur $\mathcal{H}^1(X, \mathrm{Ad}(\rho_0))$, qui permet en particulier de déduire que, si le domaine des périodes G_0/V_0 est de type Hermitien symétrique, alors la matrice hessienne est semi-définie positive et les directions le long desquelles elle est nulle sont exactement celle qui restent au premier ordre des variations de structures de Hodge.

6.1 The second variation of the energy

Definition 6.1.1. Let $\rho_t^{(2)} = (\rho_0, c, k) \colon \Gamma \to J^2G$ be a second order deformation of a representation and (f, v, w) a $\rho_t^{(2)}$-equivariant second order deformation. We define its energy as

$$E(f,v,w) = E(f) + t\int \langle \nabla v, \mathrm{d}f\rangle + \frac{t^2}{2}\int \langle \nabla w, \mathrm{d}f\rangle + \sum_{i,j} g^{ij} R^N\big(\frac{\partial f}{\partial x_i}, v, v, \frac{\partial f}{\partial x_j}\big) + \|\nabla v\|^2. \tag{6.1}$$

To see that when f_t is defined for $t \in (-\varepsilon, \varepsilon)$ this coincides, up to the second order, with $E(f_t)$, we derive with respect to t the equality $\frac{\partial E(f_t)}{\partial t} = \int \langle \nabla \frac{\partial f_t}{\partial t}, \mathrm{d}f_t\rangle$, and obtain:

$$\begin{aligned}\frac{\partial^2 E(f_t)}{\partial t^2}\Big|_{t=0} &= \int \Big\langle \frac{D}{\partial t}\nabla \frac{\partial f_t}{\partial t}\Big|_{t=0}, \mathrm{d}f\Big\rangle + \int \Big\langle \nabla \frac{\partial f_t}{\partial t}, \frac{D}{\partial t}\mathrm{d}f_t\Big\rangle\Big|_{t=0} \\ &= \int \Big\langle \nabla \frac{D}{\partial t} v, \mathrm{d}f\Big\rangle + \sum_{i,j} R^N\Big(\frac{\partial f}{\partial x_i}, v, v, \frac{\partial f}{\partial x_j}\Big) + \|\nabla v\|^2,\end{aligned}$$

where in the second equality we have made use of lemma 3.3.3.

Applying β_N to equation (6.1) and using the formula for the curvature of a symmetric space (1.12), we get the alternative expression:

$$\begin{aligned}\frac{\partial^2 E(f,v,w)}{\partial t^2}\Big|_{t=0} &= \int \big\langle \overset{\mathrm{can}}{\nabla} \beta_N(w), \tilde\beta \big\rangle + \big\langle \big[[\tilde\beta, \beta_N(v)], \beta_N(v)\big], \tilde\beta \big\rangle + \big\| \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]} \big\|^2 \\ &= \int \big\langle \overset{\mathrm{can}}{\nabla} \beta_N(w), \tilde\beta \big\rangle + \big\| [\tilde\beta, \beta_N(v)] \big\|^2 + \big\| \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]} \big\|^2 \qquad (6.2)\end{aligned}$$

Lemma 6.1.2. *The integrals in* (6.1) *are well defined.*

Proof. Up to the first order, this is just lemma 4.1.1. For the integral multiplying t^2, we will prove the good definition for the equivalent expression in (6.2). Letting Σ be a fundamental domain for the action of Γ on $\widetilde{M}$, we have to see that integrating over Σ or over $\gamma\Sigma$ gives the same result. Otherwise said, we have to substitute in (6.2) the values in $\gamma\tilde x$, according to definition 5.1.6, and check that all the extra terms cancel out. So let us see how each term of (6.2) changes under multiplication by γ:

$$\begin{aligned}\big\langle \overset{\mathrm{can}}{\nabla} \beta_N(w), \tilde\beta \big\rangle_{\gamma\tilde x} &= \big\langle \mathrm{Ad}_{\rho_0(\gamma)} \overset{\mathrm{can}}{\nabla} \beta_N(w), \tilde\beta_{\gamma\tilde x} \big\rangle + \big\langle \overset{\mathrm{can}}{\nabla} k(\gamma)^{[\mathfrak{p}]_{\gamma\tilde x}}, \tilde\beta_{\gamma\tilde x} \big\rangle \\ &+ 2\Big\langle \big[\overset{\mathrm{can}}{\nabla} c(\gamma)^{[\mathfrak{k}]_{\gamma\tilde x}}, \beta_N(v)\big], \tilde\beta_{\gamma\tilde x} \Big\rangle + 2\Big\langle \big[c(\gamma)^{[\mathfrak{k}]_{\gamma\tilde x}}, \overset{\mathrm{can}}{\nabla} \beta_N(v)\big], \tilde\beta_{\gamma\tilde x} \Big\rangle \\ &+ \Big\langle \big[\overset{\mathrm{can}}{\nabla} c(\gamma)^{[\mathfrak{k}]_{\gamma\tilde x}}, c(\gamma)^{[\mathfrak{p}]_{\gamma\tilde x}}\big], \tilde\beta_{\gamma\tilde x} \Big\rangle + \Big\langle \big[c(\gamma)^{[\mathfrak{k}]_{\gamma\tilde x}}, \overset{\mathrm{can}}{\nabla} c(\gamma)^{[\mathfrak{p}]_{\gamma\tilde x}}\big], \tilde\beta_{\gamma\tilde x} \Big\rangle.\end{aligned}$$

Since $k(\gamma)$ is constant, we have $\overset{\mathrm{can}}{\nabla} k(\gamma) = -[\tilde\beta, k(\gamma)]$, so that $\langle \overset{\mathrm{can}}{\nabla} k(\gamma), \tilde\beta_{\gamma\tilde x} \rangle = 0$, as in the proof of lemma 4.1.1; also, $\overset{\mathrm{can}}{\nabla} c(\gamma)^{[\mathfrak{p}]} = -\big[\tilde\beta, c(\gamma)^{[\mathfrak{k}]}\big]$ and $\overset{\mathrm{can}}{\nabla} c(\gamma)^{[\mathfrak{k}]} = -\big[\tilde\beta, c(\gamma)^{[\mathfrak{p}]}\big]$, since $\overset{\mathrm{can}}{\nabla}$ is compatible with projections on $[\mathfrak{k}]$ and on $[\mathfrak{p}]$, while Lie bracket with $\tilde\beta$ exchanges them. Recalling that $[\mathfrak{k}] \oplus [\mathfrak{p}]$ is the decomposition in anti-selfadjoint and selfadjoint parts, we get:

$$\begin{aligned}\big\langle \overset{\mathrm{can}}{\nabla} \beta_N(w), \tilde\beta \big\rangle_{\gamma\tilde x} &= \big\langle \mathrm{Ad}_{\rho_0(\gamma)} \overset{\mathrm{can}}{\nabla} \beta_N(w), \tilde\beta \big\rangle - 2\Big\langle \big[\tilde\beta, c(\gamma)^{[\mathfrak{p}]}\big], \big[\tilde\beta, \mathrm{Ad}_{\rho_0(\gamma)} \beta_N(v)\big] \Big\rangle \\ &+ 2\Big\langle \mathrm{Ad}_{\rho_0(\gamma)} \overset{\mathrm{can}}{\nabla} \beta_N(v), \big[\tilde\beta, c(\gamma)^{[\mathfrak{k}]}\big] \Big\rangle - \Big\langle \big[\tilde\beta, c(\gamma)^{[\mathfrak{p}]}\big], \big[\tilde\beta, c(\gamma)^{[\mathfrak{p}]}\big] \Big\rangle \\ &- \Big\langle \big[\tilde\beta, c(\gamma)^{[\mathfrak{k}]}\big], \big[\tilde\beta, c(\gamma)^{[\mathfrak{k}]}\big] \Big\rangle.\end{aligned}$$

For the other two terms in (6.2), we obtain through analogous but easier computations:

$$\begin{aligned}\big\| [\tilde\beta, \beta_N(v)] \big\|^2_{\gamma\tilde x} &= \big\| [\tilde\beta, \beta_N(v)] \big\|^2_{\tilde x} + \big\| [\tilde\beta, c(\gamma)^{[\mathfrak{p}]}] \big\|^2 + 2\Big\langle \big[\tilde\beta, \beta_N(v)\big], \big[\tilde\beta, c(\gamma)^{[\mathfrak{p}]}\big] \Big\rangle; \\ \big\| \overset{\mathrm{can}}{\nabla} \beta_N(v) \big\|^2_{\gamma\tilde x} &= \big\| \overset{\mathrm{can}}{\nabla} \beta_N(v) \big\|^2_{\tilde x} + \big\| \big[\tilde\beta, c(\gamma)^{[\mathfrak{k}]}\big] \big\|^2 - 2\Big\langle \overset{\mathrm{can}}{\nabla} \beta_N(v), \big[\tilde\beta, c(\gamma)^{[\mathfrak{k}]}\big] \Big\rangle.\end{aligned}$$

Adding the three expressions together, the "extra terms" cancel out, giving the desired result. □

Proposition 6.1.3. *Let $\rho_t^{(2)} = (\rho_0, c, k)\colon \pi_1(M) \to G$ be a second order deformation of ρ_0, $f\colon \tilde{M} \to N$ a harmonic metric, and suppose that $(v, w) = \vartheta_{J^2N}(F, F_2)$ as in theorem 5.7.1. Let $\psi = \mathrm{d}F_2 + [\omega, F]$. Then, we have the following expression for the second variation of the energy $E_t = E(f_t)$:*

$$\frac{\partial^2 E_t}{\partial t^2}\bigg|_{t=0} = \int_M \big\langle \psi, \beta \big\rangle + \|\omega^{[\mathfrak{p}]}\|^2. \tag{6.3}$$

Proof. Recall that the relation between (v, w) and (F, F_2) is:

$$\beta_N(v) = F^{[\mathfrak{p}]}; \quad \beta_N(w) = F_2^{[\mathfrak{p}]} + \big[F^{[\mathfrak{k}]}, F^{[\mathfrak{p}]}\big].$$

Since $\omega = \mathrm{d}F$, by definition of $\mathrm{d}^{\mathrm{can}}$ we get

$$\begin{aligned} \omega^{[\mathfrak{p}]} &= \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]} + [\tilde{\beta}, F^{[\mathfrak{k}]}], \quad \text{hence} \\ \big\|\omega^{[\mathfrak{p}]}\big\|^2 &= \big\|\overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]}\big\|^2 + \big\|[\tilde{\beta}, F^{[\mathfrak{k}]}]\big\|^2 + 2\big\langle \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]}, [\tilde{\beta}, F^{[\mathfrak{k}]}]\big\rangle. \end{aligned}$$

Thus, comparing (6.2) to (6.3), we are reduced to proving that

$$\langle \overset{\mathrm{can}}{\nabla} \beta_N(w), \tilde{\beta}\rangle + \|[\tilde{\beta}, \beta_N(v)]\|^2 = \langle \psi, \tilde{\beta}\rangle + \|[\tilde{\beta}, F^{[\mathfrak{k}]}]\|^2 + 2\langle \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]}, [\tilde{\beta}, F^{[\mathfrak{k}]}]\rangle. \tag{6.4}$$

The relation between F_2 and ψ can be written as $\overset{\mathrm{can}}{\nabla} F_2 + [\tilde{\beta}, F_2] = \mathrm{d}F_2 = \psi + [F, \omega]$, and thanks to the above relation between (F, F_2) and (v, w) we get

$$\overset{\mathrm{can}}{\nabla} \beta_N(w) = \Big(\psi + [F, \omega] - [\tilde{\beta}, F_2]\Big)^{[\mathfrak{p}]} + [\overset{\mathrm{can}}{\nabla} F^{[\mathfrak{k}]}, F^{[\mathfrak{p}]}] + [F^{[\mathfrak{k}]}, \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]}].$$

Note that $\langle [\tilde{\beta}, F_2], \tilde{\beta}\rangle = \sum_i \langle F_2, [\tilde{\beta}(E_i), \tilde{\beta}(E_i)]\rangle = 0$. Replacing ω by the corresponding terms in $F^{[\mathfrak{k}]}$ and $F^{[\mathfrak{p}]}$ we get

$$\begin{aligned} \langle \overset{\mathrm{can}}{\nabla} \beta_N(w), \tilde{\beta}\rangle &= \big\langle \psi + \big[F^{[\mathfrak{p}]}, \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{k}]} + [\tilde{\beta}, F^{[\mathfrak{p}]}]\big] + \big[F^{[\mathfrak{k}]}, \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]} + [\tilde{\beta}, F^{[\mathfrak{k}]}]\big], \tilde{\beta}\big\rangle \\ &\quad + \big\langle [\overset{\mathrm{can}}{\nabla} F^{[\mathfrak{k}]}, F^{[\mathfrak{p}]}] + [F^{[\mathfrak{k}]}, \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]}], \tilde{\beta}\big\rangle \\ &= \big\langle \psi + 2\big[F^{[\mathfrak{k}]}, \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]}\big] + \big[F^{[\mathfrak{p}]}, [\tilde{\beta}, F^{[\mathfrak{p}]}]\big] + \big[F^{[\mathfrak{k}]}, [\tilde{\beta}, F^{[\mathfrak{k}]}]\big], \tilde{\beta}\big\rangle \\ &= \langle \psi, \tilde{\beta}\rangle + 2\langle \overset{\mathrm{can}}{\nabla} F^{[\mathfrak{p}]}, [\tilde{\beta}, F^{[\mathfrak{k}]}]\rangle - \|[\tilde{\beta}, F^{[\mathfrak{p}]}]\|^2 + \|[\tilde{\beta}, F^{[\mathfrak{k}]}]\|^2, \end{aligned}$$

which is exactly (6.4). □

Remark 6.1.4. Equation (6.3) implies that $\int_M \langle \psi, \beta \rangle$ is independent of ψ, even when it is not uniquely determined (cfr. corollary 5.4.3). We can see this directly, too: In this same corollary, we prove that if ψ' is another 1-form that can arise, then $\psi' = \psi + 2[\omega, \xi^{\mathfrak{k}}]$, where $\xi^{\mathfrak{k}} \in \mathfrak{h}^{\mathfrak{k}}$. Then

$$\int_M \langle \psi', \beta \rangle = \int \langle \psi, \beta \rangle + 2\langle \omega, [\xi^{\mathfrak{k}}, \beta] \rangle = \int_M \langle \psi, \beta \rangle,$$

as $[\xi, \beta] = 0$ for all $\xi \in \mathfrak{h}$.

Example 6.1.5. As in example 4.1.4, we want to prove that when $M = (X, \Omega)$ is a Kähler manifold the second variation of the energy is independent of the metric in its Kähler class. First, observe that the 1-forms involved are independent of the metric in its Kähler class, since they are all constructed starting from the metric on the bundle and the Hodge $*$ on X, which depends only on Ω. Then we may conclude thanks to the following lemma (which also includes the case of example 4.1.4).

Lemma 6.1.6. *Let α_1, α_2 be two $\mathfrak{g}$-valued 1-forms on a compact Kähler manifold (X, Ω), and suppose that at least one of them takes values in the subbundle $[\mathfrak{p}]$. Then their L^2 product*

$$\int_X \big\langle \alpha_1, \alpha_2 \big\rangle\, \Omega^n = \int_X \big\langle \alpha_1 \wedge *\alpha_2 \big\rangle \tag{6.5}$$

is independent of the metric chosen in the Kähler class Ω.

Proof. We may suppose that both forms take values in $[\mathfrak{p}]$, since the projection $\mathfrak{g} \to [\mathfrak{p}]$ depends only on the metric on the bundle and not on that on X. Write $\alpha_2 = \alpha_2' + \alpha_2''$ for its decomposition into $(1,0)$ and $(0,1)$ parts; then, since α_2 is self-adjoint, they are on the adjoint of the other, so we may write $\alpha_2 = \varphi + \varphi^*$, where $\varphi = \alpha_1'$. In particular, one sees immediately that we have

$$*\alpha_2 = (\varphi^* - \varphi) \wedge \Omega^{n-1}.$$

The quantity in (6.5) may then be rewritten (up to a constant) as

$$\int_X \operatorname{trace}\big(\alpha_1 \wedge (\varphi - \varphi^*)\big) \wedge \Omega^{n-1}.$$

Thus we only need to prove that $\operatorname{trace}(\alpha_1 \wedge (\varphi - \varphi^*))$ is a closed 2-form. Observe that $\varphi - \varphi^*$ takes values in the anti self-adjoint part of $\mathfrak{g} \otimes \mathbb{C}$. Recalling that $\mathrm{d} = \mathrm{d}^{\mathrm{can}} + \mathrm{ad}(\beta)$ is the decomposition into the metric and self-adjoint parts of the connection, we thus have

$$\operatorname{trace}\Big(\mathrm{d}\big(\alpha_1 \wedge (\varphi - \varphi^*)\big)\Big) = \operatorname{trace}\Big(\big[\beta, \alpha_1\big] \wedge (\varphi - \varphi^*)\Big) - \operatorname{trace}\Big(\alpha_1 \wedge \big[\beta, (\varphi - \varphi^*)\big]\Big).$$

However, this last expression vanishes: Indeed, by the cyclic symmetry of the trace, any expression of the form $\mathrm{trace}([\alpha_1, \alpha_2] \wedge \alpha_3)$ is invariant under cyclic permutations; however, since we are dealing with 1-forms, we also have $[\alpha_1, \alpha_2] = [\alpha_2, \alpha_1]$, hence the above expression is invariant under any permutation of $\{1, 2, 3\}$. For analogous reasons, it also equals $\mathrm{trace}(\alpha_1, [\alpha_2, \alpha_3])$, thus we obtain the desired cancellation. □

Example 6.1.7. We continue the example of section 3.7 to express the second order of the energy when $G = \mathbb{C}^*$. We have seen that the (real) harmonic form $\omega^{\mathfrak{p}}$ in this case is simply $\frac{\partial \tilde{\beta}_t}{\partial t}\big|_{t=0}$; thanks to the description of section 5.8 and the Lie algebra being abelian, the expression of $\psi^{\mathfrak{p}} = \omega_2^{\mathfrak{p}}$ is just as simple: $\psi^{\mathfrak{p}} = \frac{\partial^2 \tilde{\beta}_t}{\partial t^2}\big|_{t=0}$. Deriving $E(t) = \frac{1}{2}\|\tilde{\beta}_t\|^2$ twice, we have

$$\frac{\partial^2 E(t)}{\partial t^2}\Big|_{t=0} = \int \frac{\partial}{\partial t}\Big\langle \frac{\partial \tilde{\beta}}{\partial t}, \tilde{\beta}_t \Big\rangle\Big|_{t=0} = \int \Big\langle \frac{\partial^2 \tilde{\beta}_t}{\partial t^2}\Big|_{t=0}, \tilde{\beta} \Big\rangle + \Big\| \frac{\partial \tilde{\beta}_t}{\partial t}\Big|_{t=0} \Big\|^2 .$$

When $f_0 \colon \tilde{M} \to \mathbb{R}_{>0}$ is induced by a $\mathbb{C}$-VHS, then we have seen that $\tilde{\beta} = 0$, hence the first term vanishes, coherently with the fact that $\mathbb{C}$-VHS are minima of the energy (the only directions along which the energy remains constant are those along $\mathfrak{k}$, i.e., unitary deformations of the representation).

6.2 Pluri-sub harmonicity with respect to the Betti complex structure

In this section we suppose that G is a complex group to deduce that the energy is strictly pluri-subharmonic (PSH) with respect to the complex structure J_B on the Betti moduli space. To do so, we have to add to the expression in (6.3) the analogous expression in terms of ω' and ψ', the 1-forms arising from c' and k', the deformations of ρ_0 obtained by deriving with respect to $J_B(\frac{\partial}{\partial t})$.

Example 6.2.1. When $X = \Sigma$ is a Riemann surface, the energy functional is a Kähler potential for the Betti complex structure, hence it is strictly plurisubharmonic and exhaustive (that is, proper). Cfr. [Hit87], Proposition (9.1).

Definition 6.2.2. Let $\{\rho_0\}$ be a smooth point of $\mathbb{M}_B(M, \mathbb{G})$, so that the tangent space is $H^1(M, \mathrm{Ad}(\rho_0))$. Then, the Betti complex structure is given by

$$\{c'\} = J_B(\{c\}) = \{ic\} \in H^1(M, \mathrm{Ad}(\rho_0)).$$

We may write $J_B(\frac{\partial}{\partial t}) = i\frac{\partial}{\partial t}$; in particular, then, $k' = -k$.

Proposition 6.2.3. *The energy functional defines a strictly plurisubharmonic function on smooth points of the Betti moduli space* $\mathbb{M}_{\mathrm{B}}(M,G)$*. More precisely, if a deformation* f_t *of* f_0 *is induced by a harmonic 1-form* ω*, we have:*

$$\left(\frac{\partial^2}{\partial t^2} + \left(J_B\frac{\partial}{\partial t}\right)^2\right)E(f_t) = \int_M \|\omega\|^2 \mathrm{dVol}_g \geqslant 0.$$

Proof. By definition, $\omega' = i\omega$. Fix some (F, F_2) such that $(v,w) = \vartheta_{J^2N}(F,F_2)$. We deduce an expression for (F', F_2'), relative to (c', k') as in proposition 5.5.5, hence for ψ'. Namely, we have

$$\begin{pmatrix} F' \\ F_2' \end{pmatrix} = \begin{pmatrix} iF \\ -F_2 - \eta \end{pmatrix},$$

where η is a section of $\mathcal{V}$ such that $J(\eta) = 2\omega^*(\omega)$. Hence

$$\psi' = \mathrm{d}F_2' + [\omega', F'] = -\mathrm{d}F_2 - \mathrm{d}\eta - [\omega, F] = -\psi - \mathrm{d}\eta.$$

Let us now sum the expression (6.3) with the corresponding one, in terms of ω' and ψ'. Recalling that multiplication by i exchanges $[\mathfrak{p}]$ and $[\mathfrak{k}]$, and that the (real) metric on G is the real part of the hermitian one, we obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} + \left(J_B\frac{\partial}{\partial t}\right)^2\right)E(f_t) &= \int \langle \psi, \beta\rangle + \left\|\omega^{[\mathfrak{p}]}\right\|^2 + \langle -\psi - \mathrm{d}\eta, \beta\rangle + \left\|(i\omega)^{[\mathfrak{p}]}\right\|^2 \\ &= \int \left\|\omega^{[\mathfrak{p}]}\right\|^2 - \langle \mathrm{d}\eta, \beta\rangle + \left\|i\omega^{[\mathfrak{k}]}\right\|^2 = \int \|\omega\|^2,\end{aligned}$$

where for the last equality we have used that $\mathrm{d}^*\beta = 0$. This is follows from lemma 1.6.3, together with $\mathrm{d}^{\mathrm{can}*}\beta = 0$, which is exactly the harmonicity of f and $\sum_{i,j} g^{ij}[\beta(\frac{\partial}{\partial x_i}), \beta(\frac{\partial}{\partial x_j})] = 0$. □

Corollary 6.2.4. *The energy functional is plurisubharmonic on the normalization of* $\mathbb{M}_B(M,G)^{red}$*. In particular, when* M *is a Riemann surface, the energy is plurisubharmonic on the whole of* $\mathbb{M}_B(M,G)$*.*

Proof. The fact that $\mathbb{M}_B(M,G)$ is normal when M is a Riemann surface has been proved in [Sim94], section 11. To prove plurisubharmonicity on the singularities, one uses the argument in [FN80], section 3. □

Remark that, when M is a Riemann surface, the energy functional thus provides an exhaustive plurisubharmonic function, since the energy is exhaustive if and only if it is proper, which is the case for Riemann surfaces (cfr. [Hit87], section 7).

Remark 6.2.5. The formula of proposition 6.2.3 actually gives a bit more: namely, the energy functional is a Kähler potential for the Kähler form defined by the Betti complex structure and the Weil-Petersson metric.

Example 6.2.6. The energy functional is not, in general, log-psh (that is, $\log(E)$ is not psh). To see this, first derive twice $\log(E(f_t))$, thus getting

$$\frac{\partial^2}{\partial t^2}\log E\big(f_t\big) = \frac{1}{E(f)^3}\bigg(\frac{\partial^2 E(f_t)}{\partial t^2}E(f) - 2\Big(\frac{\partial E(f_t)}{\partial t}\Big)^2\bigg).$$

Summing the expression along (ω,ψ) to the one along $(i\omega,-\psi)$, as in the proof of proposition 6.2.3, one gets

$$\bigg(\frac{\partial^2}{\partial t^2}+\Big(J_B\frac{\partial}{\partial t}\Big)^2\bigg)\log E\big(f_t\big)$$
$$= \frac{1}{E(f)^3}\bigg(\int_X\|\omega\|^2\cdot\frac{1}{2}\int_X\|\beta\|^2 - 2\Big(\int_X\langle\omega,\beta\rangle\Big)^2 - 2\Big(\int_X\langle i\omega,\beta\rangle\Big)^2\bigg).$$

This is positive if, and only if,

$$\|\beta\|_{L^2}\cdot\|\omega\|_{L^2} \geqslant 2\sqrt{\big\langle\omega,\beta\big\rangle_{L^2}^2+\big\langle i\omega,\beta\big\rangle_{L^2}^2}. \tag{6.6}$$

This is strictly stronger than the Cauchy-Schwarz inequality $\|\beta\|_{L^2}\cdot\|\omega\|_{L^2} \geqslant \langle\omega,\beta\rangle_{L^2}$, so any case where one is allowed to take $\omega=\beta$ gives a counterexample to (6.6). This is the case, for example, when $G=\mathbb{C}^*$ as in section 3.7; in that case, we are given a non-trivial 1-cocycle $r\colon\Gamma\to\mathbb{R}$ and a harmonic function $g\colon\widetilde{M}\to\mathbb{R}$ such that $g(\gamma\tilde{x}) = g(\tilde{x})+r(\gamma)$. We have noticed that in this case β is a harmonic 1-form representing c, so that one can take $\omega=\beta$, for example by considering the deformation $r_t(\gamma)=e^t r(\gamma)$ and $g_t(\tilde{x})=e^t g(\tilde{x})$. Then $\langle i\omega,\beta\rangle = 0$, thus

$$\bigg(\frac{\partial^2}{\partial t^2}+\Big(J_B\frac{\partial}{\partial t}\Big)^2\bigg)\log E\big(f_t\big) = -\frac{3}{E(f_t)^2}<0.$$

6.3 Minimality of VHS with Hermitian symmetric period domain

In this section, $M=X$ will be a compact Kähler manifold, G a complex algebraic group, $\rho_0\colon\Gamma\to G$ a representation induced by a $\mathbb{C}$-VHS, G_0 the closure of its image, K_0 a maximal subgroup therein, and K a maximal subgroup of G containing K_0. Let $f\colon\widetilde{X}\to G_0/K_0\subset G/K$ be induced by the period mapping, $\rho_t^{(2)}$ a second order deformation of ρ_0 along which f is deformable in the strong sense, that is, a ψ as in definition 5.4.1 exists. We want to inquire on the positivity of the Hessian of the energy at this critical point, and to give a formula for the second derivative of the energy in terms of the eigenvalues of γ.

This problem has been intensively studied in the case where $X = \Sigma$ is a Riemann surface of genus $g > 1$. In that case, the energy functional is a perfect Morse function, so the study of its minima and Morse indices gives information on the topology of the moduli space of representations (e.g. the number of connected components equals that of the minima locus). Even in the simplest cases, if we admit complex deformations (that is, such that ρ_t quits G_0), one does not hope to obtain non-trivial minima:

Proposition 6.3.1 (Hitchin, [Hit87], (7.1))**.** *In $\mathbb{M}_{\mathrm{B}}(\Sigma, \mathbb{P}\mathrm{SL}(2,\mathbb{C}))$ the only critical value of index 0 is 0 (i.e. the local minima are indeed global ones and correspond to unitary representations). The other critical values are of the form $(d - \frac{1}{2})\pi$, with $0 < d < g - 1$, and have index $2(g + 2d - 2) > 0$.*

To analyze the analogous problem for different groups, Hitchin [Hit92], §9, proves another result relating the eigenvectors of the Hessian of the energy to those of γ, defined as in lemma 4.2.4. Recall that in the case of the Riemann surfaces, the moduli space is constructed as a quotient of the (infinite dimensional) affine space $\mathcal{A} \times \Omega$, where $\mathcal{A}$ is the space of flat connections on a principal G-bundle P and $\Omega = \Omega^{0,1}(\Sigma, \mathrm{ad}(P) \otimes \mathbb{C})$ is the space of Higgs bundles. In particular, the tangent vectors to the moduli space lift to pairs $(\dot{A}, \dot{\Phi})$; notice that, in our notations, one has

$$\dot{A} = \left(\omega^{[\mathfrak{k}]}\right)''; \quad \dot{\Phi} = \left(\omega^{[\mathfrak{p}]}\right)', \tag{6.7}$$

where $\alpha^{[\mathfrak{k}]}(\chi) = \alpha(\chi)^{[\mathfrak{k}]}$ for *real* tangent fields $\chi \in \Xi(\Sigma)$ (and similarly for $[\mathfrak{p}]$) and α' (resp. α'') stands for the holomorphic (resp. anti-holomorphic) part of a 1-form α. Then Hitchin proves (cfr. also [Got95], sec. 2.3.2, for a different proof):

Proposition 6.3.2 (Hitchin,Gothen)**.** *If the S^1 action on $(\dot{A}, \dot{\Phi})$ has weights (m, n), that is, γ acts on it with eigenvalues (im, in), then the Hessian of the energy has eigenvalues $(-m, 1 - n)$ on the image of $(\dot{A}, \dot{\Phi})$.*

This result is used for example by Hitchin, loc. cit., to study the topology of $\mathbb{M}_{\mathrm{B}}(\Sigma, \mathbb{P}\mathrm{SL}(n,\mathbb{R}))$, by Gothen for that of $\mathbb{M}_{\mathrm{B}}(\Sigma, \mathrm{Sp}(2n,\mathbb{R}))$, by Bradlow–García-Prada–Gothen [BGPG03] for that of $\mathbb{M}_{\mathrm{B}}(\Sigma, U(p,q))$, to cite a few. Our aim is to generalize this result to higher dimension. Before stating the results, we introduce some nomenclature for the different objects in this setting.

Denote by $\overline{G_0^{\mathbb{C}}}$ the complex Zariski closure of G_0. By hypothesis, ρ_0 being induced by a $\mathbb{C}$-VHS means that there is a faithful linear representation $\overline{G_0^{\mathbb{C}}} \hookrightarrow \mathrm{GL}(r, \mathbb{C})$ such that the resulting vector bundle $\mathcal{V} = (\tilde{X} \times \mathbb{C}^r)/\Gamma$ supports a $\mathbb{C}$-VHS; we give $\mathrm{End}(\mathcal{V}) = (\tilde{X} \times \mathfrak{gl}_n(\mathbb{C}))/\Gamma$ the induced $\mathbb{C}$-VHS structure of weight 0. Then we know that G_0 is the intersection of $\overline{G_0^{\mathbb{C}}}$ with the subgroup $U(p, q)$ of $\mathrm{GL}(r, \mathbb{C})$ respecting the polarization (this is essentially the content of a theorem by Karpelevich and Mostow, [Kar53, Mos55]); letting $\mathfrak{u}$ denote the Lie algebra of $U(p, q)$, if we set $\mathfrak{k}_\mathfrak{u} = \mathfrak{u} \cap \bigoplus_{p\equiv 0} \mathfrak{g}^{-p,p}$ and $\mathfrak{p}_\mathfrak{u} = \mathfrak{u} \cap \bigoplus_{p\equiv 0} \mathfrak{g}^{-p,p}$ we obtain a Cartan decomposition for $\mathfrak{u}$. We define $\mathfrak{k} = \mathfrak{k}_\mathfrak{u} \oplus i\mathfrak{p}_\mathfrak{u}$ and $\mathfrak{p} = \mathfrak{p}_\mathfrak{u} \oplus i\mathfrak{k}_\mathfrak{u}$ for the induced Cartan decomposition on $\mathfrak{g}$. Then since f takes values in G_0/K_0, every two terms of the decomposition $\mathfrak{g} = \mathfrak{k}_\mathfrak{u} \oplus \mathfrak{p}_\mathfrak{u} \oplus i\mathfrak{k}_\mathfrak{u} \oplus i\mathfrak{p}_\mathfrak{u}$ are orthogonal with respect to the metric on $\tilde{X} \times \mathfrak{g}$, which is twice the real part of the Hermitian extension $\langle \cdot, \cdot \rangle_\mathbb{C}$ of the metric $\langle \cdot, \cdot \rangle$ induced on $\mathfrak{u}$ by f. Taking an adequate faithful representation, then, we can suppose without loss of generality that $\mathfrak{g} = \mathfrak{gl}_r(\mathbb{C})$. Explicitely, for every ξ, η in $\mathfrak{g}$:

$$\langle \xi, \eta \rangle = 2\mathcal{R}e\langle \xi, \eta \rangle_\mathbb{C}. \tag{6.8}$$

The reason for the "2" here is that, according to our definitions, one has to regard $\mathfrak{g}$ as a real Lie algebra, and define the scalar product on it by the (real) Killing form one obtains. This is twice the real part of the complex Killing form on $\mathfrak{g}$ (that coincides with the complexification of the real one on $\mathfrak{g}_0$). Of course, one can modify the definitions to get rid of this coefficient, but we keep notations coherent with our preceeding chapters in order to be able to use (6.3) without modifications.

Denote σ_0 the Cartan involution associated to $\mathfrak{u} = \mathfrak{k}_\mathfrak{u} \oplus \mathfrak{p}_\mathfrak{u}$, and $\sigma_\mathbb{C}$ its $\mathbb{C}$-linear extension. Then, with respect to the Hodge decomposition of lemma 4.2.2, for every $\xi = \sum \xi^{-p,p} \in \mathfrak{g}$ one has

$$\sigma_\mathbb{C}(\xi) = \xi^{[\mathfrak{p}_\mathfrak{u} + i\mathfrak{p}_\mathfrak{u}]} - \xi^{[\mathfrak{k}_\mathfrak{u} + i\mathfrak{k}_\mathfrak{u}]} = \sum_p (-1)^{p+1} \xi^{-p,p}.$$

For every $\xi = \xi_1 + i\xi_2$, with $\xi_i \in \mathfrak{u}$, write $\bar{\xi} = \xi_1 - i\xi_2$ and $\xi^* = \xi^{[\mathfrak{p}]} - \xi^{[\mathfrak{k}]}$ (so that $\mathrm{ad}(\xi)$ is adjoint to $\mathrm{ad}(\xi^*)$ with respect to $\langle \cdot, \cdot \rangle$). Then we have:

$$\xi^* = -\sigma_\mathbb{C}(\bar{\xi}). \tag{6.9}$$

Finally, recall that on $\mathcal{H}^1(X, \mathrm{Ad}(\rho_0))$ there is the Deligne's Hodge structure of weight 1 indexed by pairs (P, Q) with $P + Q = 1$, that we write as

$$\omega^{(P,Q)} = (\omega')^{P-1,1-P} + (\omega'')^{P,-P}. \tag{6.10}$$

Lemma 6.3.3. *Let $\rho_t^{(2)} = (\rho_0, c, k)\colon \Gamma \to J^2G$ be a second order deformation of ρ_0, and (f, v, w) a second order harmonic and $\rho_t^{(2)}$-equivariant deformation of f. Suppose that X is Kähler and that f_0 is induced by a $\mathbb{C}$-VHS. Then, the second variation of the energy reads*

$$\frac{\partial^2 E(f_t)}{\partial t^2}\Big|_{t=0} = \int_X \big\langle \Lambda[\omega,\omega],\gamma\big\rangle + \|\omega^{[\mathfrak{p}]}\|^2.$$

In particular, the second order variation depends on the f and c only, indipendently of any second order data.

Proof. Thanks to the Kähler identities (cfr. [Zuc79] or [Sim92]), corollary 4.2.6 and the identity $\mathrm{d}\psi = -[\omega,\omega]$ in (5.10), we have

$$\int\langle\psi,\beta\rangle \overset{4.2.6}{=} \int\langle\psi, D^c\gamma\rangle = \int\langle D^{c*}\psi,\gamma\rangle \overset{\mathrm{KI}}{=} \int\langle -[\Lambda,\mathrm{d}]\psi,\gamma\rangle \overset{(5.10)}{=} \int\langle\Lambda[\omega,\omega],\gamma\rangle.$$

□

We are ready for the proof of theorem 6.3.10. However, in the special case where ω is only allowed to take values in $\tilde{X}\times_\Gamma \mathfrak{g}_0$, the proof simplifies significantly, and we also get an interpretation in terms of the Deligne's Hodge structure (which does not hold in the general setting). This will give a positivity result for such deformations (corollary 6.3.8). For these reasons, and to guide the reader through the longer general proof, we start by this special case.

Proposition 6.3.4. *Suppose that $\rho_0\colon \Gamma \to G$ is a representation, where G is a complex algebraic group, $G_0 = \mathrm{Image}(\rho_0)$ and ρ_0 is induced by a complex variation of Hodge structure. Let $f\colon \tilde{M} \to G_0/K_0 \subset G/K$ be the harmonic map induced by the period mapping, $\rho_t^{(2)}\colon \Gamma \to J^2G$ a second order deformation of ρ_0, and suppose that:*

- *the harmonic 1-form ω constructed from f and $\rho_t^{(1)}$ takes values in $\tilde{X}\times_\Gamma \mathfrak{g}_0$;*
- *the pair (f, c) has vanishing obstruction for the existence of ψ, that is, any of the conditions of proposition 5.4.6 is verified.*

Then, the following equivalent expressions for the second variation of the *energy hold:*

$$\frac{\partial^2 E(f_t)}{\partial t^2}\Big|_{t=0} = 2\int_X \sum_p c_p \big\|(\omega')^{-p,p}\big\|^2, \quad c_p = \begin{cases} p, & \textit{if } p \textit{ is even,} \\ 1-p, & \textit{if } p \textit{ is odd.} \end{cases}$$
$$= 2\int_X -\sum_{p\equiv 0} p\big\|(\omega'')^{-p,p}\|^2 + \sum_{p\equiv 1}(1-p)\big\|(\omega')^{-p,p}\big\|^2$$
$$= \int_X \sum_{P+Q=1} c_P \|\omega^{(P,Q)}\|^2 = \int_X \sum_{P \text{ even}} 2P\|\omega^{(P,Q)}\|^2.$$

Proof. Since all the terms appearing in 6.3.3 depend on ρ_0, f and ω only, we can and will suppose that G is the complexification of G_0.

The coordinate expression for the dual Lefschetz operator applied to $[\omega,\omega]$ gives:

$$\Lambda[\omega,\omega] = -2i\sum_j [\omega,\omega](\partial_j,\overline{\partial_j}) = -i\sum_j \big[\omega(2\partial_j),\omega(2\overline{\partial_j})\big].$$

Using (6.8), then, the first term in lemma 6.3.3 then becomes (omitting summations on j):

$$\int \big\langle \Lambda[\omega,\omega],\gamma\big\rangle = \int \mathcal{R}e\Big(-2i\Big\langle \omega(2\partial_j),\big[\gamma,\omega(2\overline{\partial_j})^*\big]\Big\rangle_{\mathbb{C}}\Big)$$
$$= \int \mathcal{R}e\Big(-2i\Big\langle \omega(2\partial_j),\big[\gamma,-\sigma_{\mathbb{C}}\big(\omega(2\partial_j)\big)\big]\Big\rangle_{\mathbb{C}}\Big),$$

where we have used (6.9) and the fact that ω is real, so that $\overline{\omega(\overline{\partial_j})} = \omega(\partial_j)$. Thus:

$$= 2\mathcal{R}e\Big(i\Big\langle \omega(2\partial_j),\sum_p \big[\gamma,(-1)^p\omega(2\partial_j)^{-p,p}\big]\Big\rangle_{\mathbb{C}}\Big)$$
$$= 2\mathcal{R}e\Big(i\sum_p\Big\langle \omega(2\partial_j)^{-p,p},(-1)^p ip\,\omega(2\partial_j)^{-p,p}\Big\rangle_{\mathbb{C}}\Big)$$
$$= 2\sum_p(-1)^p p\big\|\omega(2\partial_j)^{-p,p}\big\|^2_{\mathbb{C}} = \sum_p(-1)^p p\big\|\omega(2\partial_j)^{-p,p}\big\|^2$$
$$= 2\sum_p(-1)^p p\big\|(\omega')^{-p,p}\big\|^2.$$

As to the second term of lemma 6.3.3, first note that for every $\mathfrak{g}$-valued 1-form α one has $\|\alpha\|^2 = \|\alpha'\|^2 + \|\alpha''\|^2$, that applied to $\omega^{-p,p}$ gives:

$$\|\omega^{-p,p}\|^2 = \big\|(\omega')^{-p,p}\big\|^2 + \big\|(\omega'')^{-p,p}\big\|^2 = \big\|(\omega')^{-p,p}\big\|^2 + \big\|(\omega')^{p,-p}\big\|^2 \qquad (6.11)$$

(here we have used that, ω being real, $\omega^{-p,p}$ is conjugated to $\omega^{p,-p}$). Again since ω is real, its $[\mathfrak{p}]$-part coincides with the projections on $[\mathfrak{g}^{-p,p}]$ for odd p, so that:

$$\|\omega^{[\mathfrak{p}]}\|^2 = \sum_{p\equiv 1} \|\omega^{-p,p}\|^2 \overset{(6.11)}{=} 2\sum_{p\equiv 1} \|(\omega')^{-p,p}\|^2.$$

Adding the two expressions together gives the first line of the statement. The one in the second line follows again by switching ω' and ω'' as in equation (6.11) for even p, and noticing that p exchanges with $-p$, hence the sign changes. To see that the last expression in terms of (P,Q) bidegree coincides with the one we found, we split $\omega^{(P,Q)}$ as in (6.10), thus getting:

$$\begin{aligned}\sum_{P+Q=1} c_P \left\|\omega^{(P,Q)}\right\|^2 = {} & \sum_{P\equiv 0} P\left(\left\|(\omega')^{P-1,1-P}\right\|^2 + \left\|(\omega'')^{P,-P}\right\|^2\right) \\ & + \sum_{P\equiv 1} (1-P)\left(\left\|(\omega')^{P-1,1-P}\right\|^2 + \left\|(\omega'')^{P,-P}\right\|^2\right).\end{aligned}$$

Now, on the one hand, thanks to the real hypothesis, we have $\|(\omega'')^{P,-P}\|^2 = \|(\omega')^{-P,P}\|^2$; on the other hand, for the two terms already in ω', we simply change indexes from P to $Q = 1-P$, to get:

$$\begin{aligned} = {} & \sum_{Q\equiv 1} (1-Q)\left\|(\omega')^{-Q,Q}\right\|^2 + \sum_{P\equiv 0} P\left\|(\omega')^{-P,P}\right\|^2 + \sum_{Q\equiv 0} Q\left\|(\omega')^{-Q,Q}\right\|^2 \\ & + \sum_{P\equiv 1} (1-P)\left\|(\omega')^{-P,P}\right\|^2,\end{aligned}$$

which is exactly what we want. To get the last expression, one uses that similar symmetries entrain $\|\omega^{(P,Q)}\| = \|\omega^{(Q,P)}\|$. □

Remark 6.3.5. The second expression for the variation of the energy in proposition 6.3.4 corresponds exactly to that in proposition 6.3.2, the factor "2" being again due to our choices in the metrics involved. This is the statement we will generalize in theorem 6.3.10. We added the other expressions here because they can be thought to provide a justification in terms of the Deligne Hodge structure on $\mathcal{H}^1(X, \mathrm{Ad}(\rho_0))$ of the coefficients c_p involved, since the last line of proposition 6.3.4 may be rewritten as:

$$\frac{\partial^2 E(f_t)}{\partial t^2}\Big|_{t=0} = \int_X \sum_{P\equiv 0} P\left\|\omega^{(P,Q)}\right\|^2 + \sum_{P\equiv 1} Q\left\|\omega^{(P,Q)}\right\|^2.$$

Corollary 6.3.6. *In the moduli space $\mathbb{M}_{\mathrm{B}}(X, \mathbb{P}\mathrm{SL}(2,\mathbb{R}))$, at every critical point the Hessian of the energy is semipositive definite.*

Proof. At every $\mathbb{C}$-VHS, one has a decomposition $\mathfrak{g} = \bigoplus_p [\mathfrak{g}^{-p,p}]$ such that $\gamma \in [\mathfrak{g}^{0,0}] \neq 0$, and $\theta \in \mathcal{A}^{1,0}([\mathfrak{g}^{-1,1}])$ so that $[\mathfrak{g}^{-1,-1}] \neq 0$. Since $\dim \mathfrak{sl}_2(\mathbb{R}) = 3$, this forces

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \bigoplus_{p=-1}^{1} [\mathfrak{g}^{-p,p}].$$

Thus the P's involved in the last expression of proposition 6.3.4 are only 0 and 2. □

As a further corollary of the proposition, we now propose to prove that when $G_0 = \overline{\mathrm{Image}(\rho_0)}$ and G_0/V_0 is a Hermitian symmetric domain, then again the Hessian is semipositive definite along directions in $\mathbb{M}_{\mathrm{B}}(X, G_0)$.

Definition 6.3.7. Let $\rho_0 \colon \Gamma \to G$ be a representation induced by a $\mathbb{C}$-VHS, and denote by G_0 the monodromy of ρ_0. We say that ρ_0 is of Hermitian symmetric type if G_0/V_0 is a Hermitian symmetric domain, where $\mathrm{Lie}(V_0) = \mathfrak{g}_0 \cap [\mathfrak{g}^{0,0}]$, as in lemma 1.9.2.

This is known to be equivalent to the Hodge structure on $\mathfrak{g}_0$ of lemma 4.2.2 being reduced to $\mathfrak{g} = [\mathfrak{g}^{-1,1}] \oplus [\mathfrak{g}^{0,0}] \oplus [\mathfrak{g}^{1,-1}]$.

Corollary 6.3.8. *Let ρ_0 be induced by a $\mathbb{C}$-VHS, and G_0 be the (real) Zariski closure of its image. If ρ_0 is of Hermitian type, then the Hessian is semipositive definite along directions in $\mathbb{M}_{\mathrm{B}}(X, G_0)$; the directions along which it vanishes are exactly those along which the deformation f_t remains $\mathbb{C}$-VHS to the first order.*

Before proving the corollary, let us explain what we mean by "remaining $\mathbb{C}$-VHS to the first order". Recall that a holomorphic map $\Phi_0 \colon \tilde{X} \to G_0/V_0$ comes from a $\mathbb{C}$-VHS if and only if it is horizontal. In our terms, this means asking that θ, the $(1,0)$-part of β, takes values in $[\mathfrak{g}^{-1,1}]$.

Definition 6.3.9. Let (f, v) be a first order deformation of a harmonic, ρ_0-equivariant f, which we suppose to be induced from a $\mathbb{C}$-VHS. Then (f, v) is said to be $\mathbb{C}$-VHS to the first order if

$$\partial\beta(v) \in \mathcal{A}^{1,0}\big([\mathfrak{g}^{-1,1}]\big).$$

Some justification is in order. Suppose that ρ_t and f_t are defined and smooth for $t \in (-\varepsilon, \varepsilon)$; then, pulling back the Maurer-Cartan form, we obtain a 1-form β_t on $\tilde{X} \times (-\varepsilon, \varepsilon)$, and we let θ_t be its $(1,0)$-part (for every fixed t). Then, if f_t is induced by a VHS for every t, we have $\theta_t \in \mathcal{A}^{1,0}([\mathfrak{g}^{-1,1}])$. The

1-form β_t verifies the Maurer-Cartan equation in both tangent vectors to $\tilde{X}$ and to $(-\varepsilon, \varepsilon)$, so that

$$0 = \mathrm{d}^{\mathrm{can}}\beta_t\Big(\frac{\partial}{\partial t}, X\Big) \implies \overset{\mathrm{can}}{\nabla}_X \beta_t\Big(\frac{\partial}{\partial t}\Big) = \frac{D}{\partial t}\beta_t(X).$$

Now $\beta_t(\frac{\partial}{\partial t}) = \beta(v)$. Considering vectors X of type $(1,0)$, we obtain that

$$\partial\beta(v) = \frac{D}{\partial t}\theta_t \in \mathcal{A}^{1,0}\big([\mathfrak{g}^{-1,1}]\big)$$

Proof of corollary 6.3.8. If ρ_0 is of Hermitian symmetric type, so that the only p appearing in $[\mathfrak{g}^{-p,p}]$ are ± 1 and 0, the last expression in proposition 6.3.4 becomes simply

$$\frac{\partial^2 E(f_t)}{\partial t^2}\Big|_{t=0} = \int_X 4\big\|(\omega')^{1,-1}\big\|^2.$$

We prove that the vanishing of this terms is equivalent to asking that (f, v) is $\mathbb{C}$-VHS to the first order. Since for every X tangent vector, $\omega(X) = \overset{\mathrm{can}}{\nabla}_X F + [\beta(X), F]$, we have:

$$\omega(\partial_j) = \partial_j F + [\theta(\partial_j), F].$$

The second summand is necessarily in $[\mathfrak{g}^{-1,1}] \oplus [\mathfrak{g}^{0,0}]$, so it does not affect $\omega(\partial_j)^{1,-1}$. Hence, the metric connection ∂ being compatible with the Hodge decomposition, and since $F^{[\mathfrak{p}]} = F^{[\mathfrak{p}_0]} = F^{-1,1} + F^{1,-1}$,

$$\omega(\partial_j)^{1,-1} = \partial F^{1,-1} = 0 \iff \frac{D}{\partial t}\theta_t\Big|_{t=0} = \partial F^{[\mathfrak{p}]} \in \mathcal{A}^{1,0}([\mathfrak{g}^{-1,1}]).$$

□

Theorem 6.3.10. *Let G be a complex algebraic group, and $\rho_0\colon \Gamma \to G$ a representation induced by a complex variation of Hodge structure. Denote by G_0 the real Zariski closure of ρ_0, by $f\colon \tilde{M} \to G_0/K_0 \subset G/K$ the harmonic map induced by the period mapping, and by $\rho_t^{(2)}\colon \Gamma \to J^2G$ a second order deformation of ρ_0 for which a ψ exists. Then, denoting by $(\dot{A}, \dot{\Phi})$ the tangent direction in the moduli space as in (6.7), the second variation of the energy along $\rho_t^{(2)}$ is given by:*

$$\frac{\partial^2 E(f_t)}{\partial t^2}\Big|_{t=0} = 2\int_X \sum_p -p\big\|\dot{A}^{-p,p}\big\|^2 + (1-p)\big\|\dot{\Phi}^{-p,p}\big\|^2. \qquad (6.12)$$

Proof. As the proof is rather long and computational, we introduce some shorter notation. For an orthonormal local system of coordinates $\frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial y_j} = i\frac{\partial}{\partial x_j}$, we write:

$$\omega\big(\frac{\partial}{\partial x_j}\big) = \xi_1 + i\xi_2; \quad \omega\big(\frac{\partial}{\partial y_j}\big) = \eta_1 + i\eta_2, \quad \xi_1, \xi_2, \eta_1, \eta_2 \in \mathfrak{u}$$

(here we drop the j in the notation to lighten it). Then we define ξ_1^p, ξ_2^p, etc., as the projection to $[\mathfrak{g}^{-p,p}]$ of ξ_1, ξ_2, etc. (remark that those live only in $\mathfrak{g}$ and no more in $\mathfrak{u}$). Denoting $\partial_j = \frac{1}{2}\frac{\partial}{\partial x_j} - \frac{i}{2}\frac{\partial}{\partial y_j}$ as usual, one gets immediately:

$$\omega(2\partial_j) = \xi_1 + \eta_2 + i\big(\xi_2 - \eta_1\big), \quad \omega(2\bar{\partial}_j) = \xi_1 - \eta_2 + i\big(\xi_2 + \eta_1\big).$$

The proof consists in showing that both the expression of lemma 6.3.3 and equation (6.12) reduce to the following expression:

$$\begin{aligned}\int_X \sum_p (-1)^p 4p\mathcal{I}m\Big(\big\langle \xi_2^p, \eta_2^p\big\rangle_{\mathbb{C}} - \big\langle \xi_1^p, \eta_1^p\big\rangle_{\mathbb{C}}\Big) &+ \sum_{p\equiv 1} 2\|\xi_1^p\|_{\mathbb{C}}^2 + 2\|\eta_1^p\|_{\mathbb{C}}^2 \\ &+ \sum_{p\equiv 0} 2\|\xi_2^p\|_{\mathbb{C}}^2 + 2\|\eta_2^p\|_{\mathbb{C}}^2.\end{aligned} \tag{6.13}$$

Let us start from the expression in lemma 6.3.3; we claim that the term involving the imaginary part in (6.13) equals the first term in the lemma, while terms involving squared normes in (6.13) correspond to $\|\omega^{[\mathfrak{p}]}\|^2$. The latter identification is easy, since

$$\begin{aligned}\|\omega^{[\mathfrak{p}]}\|^2 &= \Big\|\omega\big(\frac{\partial}{\partial x_j}\big)^{[\mathfrak{p}]}\Big\|^2 + \Big\|\omega\big(\frac{\partial}{\partial y_j}\big)^{[\mathfrak{p}]}\Big\|^2 \\ &= \|\xi_1^{[\mathfrak{p}_\mathfrak{u}]}\|^2 + \|(i\xi_2)^{[i\mathfrak{k}_\mathfrak{u}]}\|^2 + \|\eta_1^{[\mathfrak{p}_\mathfrak{u}]}\|^2 + \|(i\eta_2)^{[i\mathfrak{k}_\mathfrak{u}]}\|^2.\end{aligned}$$

Now one simply uses that $[\mathfrak{p}_\mathfrak{u} + i\mathfrak{p}_\mathfrak{u}] = \bigoplus_{p\equiv 1}[\mathfrak{g}^{-p,p}]$ and that $[\mathfrak{k}_\mathfrak{u} + i\mathfrak{k}_\mathfrak{u}] = \bigoplus_{p\equiv 0}[\mathfrak{g}^{-p,p}]$ to get the result. As for the first summand, we start as in the proof of proposition 6.3.4, but here things are more complicated as ω does not commute with conjugation. We obtain:

$$\begin{aligned}\int_X \big\langle \Lambda[\omega,\omega],\gamma\big\rangle_{\mathbb{C}} &= \int_X \mathcal{R}e\Big(-2i\Big\langle \omega(2\partial_j), \big[\gamma, \omega(2\overline{\partial_j})^*\big]\Big\rangle\Big) \\ &= \int_X \mathcal{R}e\Big(-2i\Big\langle \xi_1 + \eta_2 + i\big(\xi_2 - \eta_1\big), \big[\gamma, \big(\xi_1 - \eta_2 + i\big(\xi_2 + \eta_1\big)\big)^*\big]\Big\rangle_{\mathbb{C}}\Big).\end{aligned} \tag{6.14}$$

First of all, recall that multiplication by i anti-commutes with adjunction, so that $(\xi_1 - \eta_2 + i(\xi_2 + \eta_1))^* = \xi_1^* - \eta_2^* - i\xi_2^* - i\eta_1^*$. Then, disregarding the purely imaginary terms, we obtain:

$$(6.14) = 2\int_X \big\langle \xi_1 + \eta_2, \big[\gamma, \xi_2^* + \eta_1^*\big]\big\rangle_{\mathbb{C}} + \big\langle \xi_2 - \eta_1, \big[\gamma, \xi_1^* - \eta_2^*\big]\big\rangle_{\mathbb{C}}.$$

Remark that these quantities are, in fact, real, since every object involved is in $\mathfrak{g}_0$. Furthermore, since ξ_1 (and ξ_2, etc.) is real, the adjunction and Hodge decomposition are compatible, that is:

$$\xi_1^* = \sum_p (-1)^{p+1} \xi_1^p, \quad \xi_2^* = \sum_p (-1)^{p+1} \xi_2^p, \quad \text{etc.}$$

Thus, by the definition of γ, we have $[\gamma, \xi_2^*] = \sum_p (-1)^{p+1} ip\xi_2^p$ (and similarly for the others), and we get:

$$\begin{aligned}(6.14) = 2\int_X \sum_p (-1)^p ip\Big(&\big\langle \xi_1^p, \xi_2^p\big\rangle_{\mathbb{C}} + \big\langle \eta_2^p, \xi_2^p\big\rangle_{\mathbb{C}} + \big\langle \xi_1^p, \eta_1^p\big\rangle_{\mathbb{C}} + \big\langle \eta_2^p, \eta_1^p\big\rangle_{\mathbb{C}}\Big) \\ &+ \sum_p (-1)^p ip\Big(\big\langle \xi_2^p, \xi_1^p\big\rangle_{\mathbb{C}} - \big\langle \xi_2^p, \eta_2^p\big\rangle_{\mathbb{C}} - \big\langle \eta_1^p, \xi_1^p\big\rangle_{\mathbb{C}} + \big\langle \eta_1^p, \eta_2^p\big\rangle_{\mathbb{C}}\Big).\end{aligned}$$

Now recall that all of this must be real, so we are really only interested in the imaginary parts of the hermitian product in this equation. Since $\big\langle \xi_1^p, \xi_2^p\big\rangle_{\mathbb{C}} + \big\langle \xi_2^p, \xi_1^p\big\rangle_{\mathbb{C}}$ is real (and similarly for η's), they cancel out. The other terms are summed up, and we obtain exactly the first half of (6.13).

It remains to prove that (6.13) equals (6.12). To do that, we need to compute the norms of $\dot{A}^{-p,p}$ and $\dot{\Phi}^{-p,p}$ in terms of ξ_1^p, etc. For example for the first one, we have

$$\begin{aligned}\big\|\dot{A}^{-p,p}\big\|^2 &= \big\|\omega^{[\mathfrak{k}]}(2\bar{\partial}_j)^{-p,p}\big\|_{\mathbb{C}}^2 = \Big\|\Big(\omega\big(\frac{\partial}{\partial x_j}\big)^{[\mathfrak{k}]} + i\omega\big(\frac{\partial}{\partial y_j}\big)^{[\mathfrak{k}]}\Big)^{-p,p}\Big\|_{\mathbb{C}}^2 \\ &= \Big\|\Big(\xi_1^{[\mathfrak{k}_\mathrm{u}]} + i\xi_2^{[\mathfrak{p}_\mathrm{u}]} + i\big(\eta_1^{[\mathfrak{k}_\mathrm{u}]} + i\eta_2^{[\mathfrak{p}_\mathrm{u}]}\big)\Big)\Big\|_{\mathbb{C}}^2 \\ &= \Big\|\Big(\xi_1^{[\mathfrak{k}_\mathrm{u}]} - \eta_2^{[\mathfrak{p}_\mathrm{u}]} + i\xi_2^{[\mathfrak{p}_\mathrm{u}]} + i\eta_1^{[\mathfrak{k}_\mathrm{u}]}\Big)\Big\|_{\mathbb{C}}^2 \\ &= \sum_{p\equiv 0} \big\|\xi_1^p + i\eta_1^p\big\|_{\mathbb{C}}^2 + \sum_{p\equiv 1} \big\|i\xi_2^p - \eta_2^p\big\|_{\mathbb{C}}^2.\end{aligned}$$

Similarly, one obtains:

$$\big\|\dot{\Phi}^{-p,p}\big\|^2 = \sum_{p\equiv 0} \big\|i\xi_2^p + \eta_2^p\big\|_{\mathbb{C}}^2 + \sum_{p\equiv 1} \big\|\xi_1^p - i\eta_1^p\big\|_{\mathbb{C}}^2.$$

Thus we can compute (6.12):

$$(6.12) = 2\sum_{p\equiv 0} -p\left\|\xi_1^p + i\eta_1^p\right\|_{\mathbb{C}}^2 + (1-p)\left\|i\xi_2^p + \eta_2^p\right\|_{\mathbb{C}}^2 \\ + 2\sum_{p\equiv 1} -p\left\|i\xi_2^p - \eta_2^p\right\|_{\mathbb{C}}^2 + (1-p)\left\|\xi_1^p - i\eta_1^p\right\|_{\mathbb{C}}^2$$

We develop the identity further, using that for any $a, b \in \mathfrak{g}$, $\|a + ib\|_{\mathbb{C}}^2 = \|a\|_{\mathbb{C}}^2 + \|b\|_{\mathbb{C}}^2 + 2\mathcal{I}m\langle a, b\rangle_{\mathbb{C}}$:

$$(6.12) = 2\sum_{p\equiv 0} -p\Big(\left\|\xi_1^p\right\|_{\mathbb{C}}^2 + \left\|\eta_1^p\right\|_{\mathbb{C}}^2 + 2\mathcal{I}m\big\langle \xi_1^p, \eta_1^p\big\rangle_{\mathbb{C}}\Big) \\ + (1-p)\Big(\left\|\xi_2^p\right\|_{\mathbb{C}}^2 + \left\|\eta_2^p\right\|_{\mathbb{C}}^2 + 2\mathcal{I}m\big\langle \eta_2^p, \xi_2^p\big\rangle_{\mathbb{C}}\Big) \\ + 2\sum_{p\equiv 1} -p\Big(\left\|\xi_2^p\right\|_{\mathbb{C}}^2 + \left\|\eta_2^p\right\|_{\mathbb{C}}^2 + 2\mathcal{I}m\big\langle \xi_2^p, \eta_2^p\big\rangle_{\mathbb{C}}\Big) \\ + (1-p)\Big(\left\|\xi_1^p\right\|_{\mathbb{C}}^2 + \left\|\eta_1^p\right\|_{\mathbb{C}}^2 + 2\mathcal{I}m\big\langle \eta_1^p, \xi_1^p\big\rangle_{\mathbb{C}}\Big)$$

Finally, we notice that ξ_1, etc., being real, $\|\xi_1^p\|_{\mathbb{C}}^2 = \|\xi_1^{-p}\|_{\mathbb{C}}^2$, etc., so that all such norms multiplied by p cancel out with the ones multiplied by $-p$. For the same reason, the terms involving, for example, $\mathcal{I}m\langle \xi_2^p, \eta_2^p\rangle_{\mathbb{C}}$ cancel out unless they are multiplied by p, since:

$$\sum_p \mathcal{I}m\big\langle \xi_2^p, \eta_2^p\big\rangle_{\mathbb{C}} = \sum_{p\geqslant 0} \mathcal{I}m\Big(\big\langle \xi_2^p, \eta_2^p\big\rangle_{\mathbb{C}} + \big\langle \xi_2^{-p}, \eta_2^{-p}\big\rangle_{\mathbb{C}}\Big) \\ = \sum_{p\geqslant 0} \mathcal{I}m\Big(\big\langle \xi_2^p, \eta_2^p\big\rangle_{\mathbb{C}} + \overline{\big\langle \xi_2^p, \eta_2^p\big\rangle_{\mathbb{C}}}\Big) = 0$$

(the term involving ξ_2^0 vanishes since ξ_2^0 and η_2^0 are in $\mathfrak{g}_0$). We are left with:

$$(6.12) = 2\sum_{p\equiv 0} \left\|\xi_2^p\right\|_{\mathbb{C}}^2 + \left\|\eta_2^p\right\|_{\mathbb{C}}^2 + 2\sum_{p\equiv 1} \left\|\xi_1^p\right\|_{\mathbb{C}}^2 + \left\|\eta_1^p\right\|_{\mathbb{C}}^2 \\ + 4\sum_{p\equiv 0} p\mathcal{I}m\Big(\big\langle \eta_1^p, \xi_1^p\big\rangle_{\mathbb{C}} + \big\langle \xi_2^p, \eta_2^p\big\rangle_{\mathbb{C}}\Big) + 4\sum_{p\equiv 1} p\mathcal{I}m\Big(\big\langle \eta_2^p, \xi_2^p\big\rangle_{\mathbb{C}} + \big\langle \xi_1^p, \eta_1^p\big\rangle_{\mathbb{C}}\Big),$$

which is exactly (6.13). □

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