Deformations of twisted harmonic maps

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Table of contents

Twisted harmonic maps

- X: closed connected Riemannian (later: Kähler) manifold;
- $\Gamma = \pi_1(X, x_0)$: its fundental group;
- $\rho\colon \Gamma \to G$: representation to an algebraic reductive group G (usually: $G = \mathrm{GL}(r, \mathbb{C})$);
- $N = G/K$: symmetric space of the non-compact type ($K = \mathrm{U}(r)$, N: positive definite Hermitian matrices);
- $\tilde{X} \to X$: universal cover.

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- $\tilde{X} \to X$: universal cover.

Definition

- A map $f\colon \tilde{X} \to N$ is ρ-equivariant (or: twisted) if $f(\gamma\tilde{x}) = \rho(\gamma)f(\tilde{x})$ for $\gamma \in \Gamma$, $\tilde{x} \in \tilde{X}$.
- It is *harmonic* if it minimizes

$$E(f) = \frac{1}{2}\int_X \|\mathrm{d}f\|^2 \mathrm{dVol}_g.$$

Harmonic metrics

Definition

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Equivariant map $f\colon \tilde{X} \to \mathrm{GL}(r,\mathbb{C})/U(r)$ $\longleftrightarrow$ Hermitian positive definite metric on $(\mathcal{V}, D) = \Big((\tilde{X} \times \mathbb{C}^r)/\Gamma, d\Big)$.

Definition

The Hermitian metric is *harmonic* if f is.

Maurer-Cartan form

Write $\mathrm{d} = \mathrm{d}^{\mathrm{can}} + \beta$, where β is a $\mathfrak{g} = \mathrm{Lie}(G)$-valued 1-form and:

- $\mathrm{d}^{\mathrm{can}}$ is a metric connection;
- β is self-adjoint. We call it the "Maurer-Cartan" 1-form.

Then $\beta \cong \mathrm{d}f$ through the identification:

$$\begin{aligned} \vartheta_{TN} \colon N \times \mathfrak{g} \supseteq [\mathfrak{p}] &\xrightarrow{\sim} TN \\ (n, \xi) &\longmapsto \frac{\partial}{\partial t}\left(\exp(t\xi) \cdot n\right)\Big|_{t=0}. \end{aligned}$$

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Theorem (Siu '80, Sampson '86)

If X is Kähler, decompose $(1,0)$ and $(0,1)$ parts:

$$\mathrm{d}^{\mathrm{can}} = \partial + \bar{\partial}; \quad \beta = \theta + \theta^*.$$

Then $\bar{\partial}^2 = 0$, $\bar{\partial}(\theta) = 0$ and $\theta \wedge \theta = 0$, that is, $(\mathcal{V}, \bar{\partial}, \theta)$ is a Higgs bundle.

Moduli spaces

Theorem (Hitchin '87, Simpson '94)

Assume further that X is projective. There exist homeomorphic moduli spaces:

$$\mathbb{M}_{\mathrm{B}}(X, G) = \mathrm{Hom}(\Gamma, G)/\!/G \xrightarrow{\sim} \mathbb{M}_{\mathrm{Dol}}(X, G).$$

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Definition

Energy functional $E\colon \mathrm{Hom}(\Gamma, G) \to \mathbb{R}$:

$$E(\rho) = \inf\left\{ E(f) = \frac{1}{2}\int_X \|\mathrm{d}f\|^2 \;\middle|\; f \text{ is } \rho\text{-equivariant, smooth} \right\}.$$

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$$E(\mathcal{E}, \theta) = \|\theta\|_{L^2}^2.$$

Motivation

Let Σ be a Riemann surface, X be Kähler.

Hitchin '87: Topology of the connected components of $\mathbb{M}_{\mathrm{B}}(\Sigma, \mathbb{P}\mathrm{SL}(2, \mathbb{R}))$.

Hitchin '92: "Teichmüller" component of $\mathbb{M}_{\mathrm{B}}(\Sigma, \mathbb{P}\mathrm{SL}(n, \mathbb{R}))$.

Bradlow–García-Prada–Gothen '03: Connected components of $\mathbb{M}_{\mathrm{B}}(\Sigma, \mathbb{P}\mathrm{U}(p, q))$.

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Toledo '12: Plurisubharmonicity of the energy on the Teichmüller space (i.e. fixed ρ but varying complex structure on Σ).
Biswas-Schumacher '06: Computation of $\partial\bar{\partial}E$ on the moduli spaces of *stable* Higgs bundles on X (i.e. where H is trivial).

The universal twisted harmonic map

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Definition

$$\tilde{x}_0 \in \tilde{X}; \quad Y = \left\{ \begin{array}{l|l} (n,\rho) \in N \times \mathrm{Hom}(\Gamma, G) & \exists f_{(n,\rho)} : \tilde{X} \to N, \\ \multicolumn{2}{l}{\rho\text{-equivariant and harmonic s.t. } f(\tilde{x}_0) = n} \end{array} \right\}.$$

Define the *universal twisted harmonic map* $\mathscr{H} : Y \times \tilde{X} \to N$ by $\mathscr{H}(n, \rho, \tilde{x}) = f_{(n,\rho)}(\tilde{x})$.

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The energy functional $E \colon \mathrm{Hom}(\Gamma, G) \to \mathbb{R}$ *is continuous.*

Basic definitions

Definition

- Deformation of ρ_0 is $c \in Z^1(\Gamma, \mathfrak{g})$; write $\rho_t^{(1)} = (\rho_0, c)$.
- Deformation of f is $v \in \mathcal{C}^\infty(f^* TN)$.

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Definition

A first order deformation $v \in \mathcal{C}^\infty(f^* TN)$ is said:

- $\rho_t^{(1)}$-equivariant if $v(\gamma\tilde{x}) = \rho_0(\gamma)v(\tilde{x}) + \vartheta_{TN}(f(\gamma\tilde{x}), c(\gamma))$;
- harmonic if $\mathcal{J}(v) = 0$, where locally

$$\mathcal{J}(v) = -\sum_{j,k} g^{jk}\left(\frac{D}{\partial x_j}\frac{D}{\partial x_k}v + R^N\left(\frac{\partial f}{\partial x_k}, v\right)\frac{\partial f}{\partial x_j}\right).$$

Fact: They are the “natural” definitions.

Construction of equivariant harmonic deformations

Let $\omega \in \mathcal{H}^1(X, \mathrm{Ad}(\rho_0))$ be harmonic, representing c.

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Theorem A

Let $\mathscr{P} = \left\{ F \colon \tilde{X} \to \mathfrak{g} \mid \mathrm{d}F = \omega, F(\gamma \tilde{x}) = \mathrm{Ad}_{\rho_0(\gamma)} F(\tilde{x}) + c(\gamma) \right\}$.
Then $\mathscr{P}$ is an affine space over $\mathfrak{h}$, the Lie algebra of $H = Z_G(\mathrm{Image}(\rho_0))$, and

$$\vartheta_{TN} \colon \mathscr{P} \twoheadrightarrow \left\{ v \in \mathcal{C}^\infty(f^* TN) \;\middle|\; \rho_t^{(1)}\text{-equivariant and harmonic} \right\}.$$

The latter space is affine over $\mathfrak{h} \cap \mathfrak{p}$ (in particular, it is always non-empty).

Critical points as $\mathbb{C}$-VHS

Corollary

If $v = \frac{\partial f_t}{\partial t}$ *is harmonic and* $\rho_t^{(1)}$*-equivariant, with* ω *as above:*

$$\frac{\partial E(f_t)}{\partial t}\Big|_{t=0} = \int_X \langle \omega, \beta \rangle \mathrm{dVol}.$$

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Theorem B

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Proof.

- $\supseteq$ S^1-action: $\varphi\colon S^1 \to G$. Let $\gamma = \frac{\partial \varphi(e^{i\theta})}{\partial \theta}\big|_{\theta=0}$. Then $\beta = D^c\gamma$.
- $\subseteq$ $\omega = \theta + D''\eta$; $0 = \int_X \langle \omega, \theta + \theta^* \rangle = \int_X \|\omega\|^2$.

Definitions

Definition

- A deformation of (ρ_0, c) is a $(c, k) \in Z^1(\Gamma, \mathrm{Ad}(\rho_0, c))$; write $\rho_t^{(2)} = (\rho_0, c, k) \colon \Gamma \to J^2 G$.
- A deformation of (f, v) is $w \in \mathcal{C}^\infty(f^* TN)$.

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Definition

$$D_2 = \begin{pmatrix} \mathrm{d} & 0 \\ \mathrm{ad}(\omega) & \mathrm{d} \end{pmatrix}; \quad D_{2,*} = \begin{pmatrix} \mathrm{d}^* & 0 \\ \omega^* \lrcorner & \mathrm{d}^* \end{pmatrix}.$$

A map $(F, F_2) \colon \tilde{X} \to \mathfrak{g} \times \mathfrak{g}$ is *equivariant of harmonic type* if

$$D_{2,*} D_2 \begin{pmatrix} F \\ F_2 \end{pmatrix} = 0; \qquad (f, F, F_2) \text{ is } \rho_t^{(2)}\text{-equivariant.}$$

Existence of (F, F_2)

Lemma

There is a $\vartheta_{J^2N}\colon N \times \mathfrak{g} \times \mathfrak{g} \to TN \times TN$ *sending equivariant* (F, F_2) *of harmonic type to equivariant harmonic* (v, w)*.*

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Definition

If (F, F_2) is equivariant of harmonic type:

$$\begin{pmatrix} \omega \\ \psi \end{pmatrix} := D_2 \begin{pmatrix} F \\ F_2 \end{pmatrix}; \quad \text{then} \quad \begin{cases} \mathrm{d}\psi = -[\omega, \omega]; \\ \mathrm{d}^*\psi = -\omega^* \lrcorner\, \omega. \end{cases}$$

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Theorem (Goldman-Millson '88)

The second order obstruction of extending a first order deformation $\rho_t^{(1)}$ *of a semi-simple* ρ_0 *is the cohomology class of* $[\omega, \omega] \in Z^2(\Gamma, \mathrm{Ad}(\rho_0))$. *If* Γ *is a Kähler group, then this is the only obstruction.*

The construction theorem

Theorem C

Let G be complex. The following are equivalent:

1. There exists an equivariant (F, F_2) of harmonic type;
2. There exists a ψ such that $\mathrm{d}\psi = -[\omega, \omega]$ and $\mathrm{d}^*\psi = -\omega^* \lrcorner\, \omega$;
3. ω is a minimum for $\|\cdot\|_{L^2}$ in its H-orbit, where $H = Z_G(\mathrm{Image}(\rho_0))$ acts on $\mathcal{H}^1(M, \mathrm{Ad}(\rho_0))$ by conjugation;
4. There exist two harmonic deformations (v, w) and (v', w'), one (ρ_0, c, k)-equivariant and the other $(\rho_0, ic, -k)$-equivariant.

Any of these is true for all harmonic metrics f if and only if $H^0(X, \mathrm{Ad}(\rho_t^{(1)}))$ is a flat $\mathbb{R}[t]/(t^2)$-module.

Plurisubharmonicity

Corollary

$$\frac{\partial^2 E(f_t)}{\partial t^2}\Big|_{t=0} = \int_X \left(\langle \psi, \beta \rangle + \|\omega^{[\mathfrak{p}]}\|^2\right) \mathrm{dVol}.$$

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Theorem D

Let G be complex, X Riemannian. The energy functional is strictly plurisubharmonic with respect to the Betti complex structure J_B. More precisely:

$$\left(\frac{\partial^2}{\partial t^2} + \left(J_B \frac{\partial}{\partial t}\right)^2\right) E(f_t) = \int_X \|\omega\|^2 \mathrm{dVol}.$$

Positivity of the Hessian

Theorem E

Suppose that X is Kähler and ρ_0 is the monodromy of a $\mathbb{C}$-VHS. Write $\dot{A} = \left(\omega^{[\mathfrak{k}]}\right)^{0,1}$ and $\dot{\Phi} = \left(\omega^{[\mathfrak{p}]}\right)^{1,0}$. Then:

$$\frac{\partial^2 E(\rho_t)}{\partial t^2}\Big|_{t=0} = 2\int_X \sum_p \Big(-p\big\|\dot{A}^{-p,p}\big\|^2 + (1-p)\big\|\dot{\Phi}^{-p,p}\big\|^2\Big)\mathrm{dVol},$$

where $\xi = \sum_p \xi^{-p,p}$ is $\mathbb{C}$-VHS of weight 0 on $\mathrm{End}(\mathcal{V}) = \tilde{X} \times_\Gamma \mathfrak{g}$.

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Corollary

If furthermore G_0/K_0 is Hermitian symmetric, then Hess$(E) \geq 0$, vanishing along $\mathbb{C}$-VHS only.